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Systems of Riemann–Liouville fractional equations with multi-point boundary conditions

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ABSTRACT

We study the existence and multiplicity of positive solutions for a system of nonlinear Riemann–Liouville fractional differential equations, subject to multi-point boundary conditions which contain fractional derivatives. The nonsingular and singular cases are investigated.

1. Introduction

We consider the system of nonlinear ordinary fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t),v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\beta}v(t) + g(t,u(t),v(t)) = 0, & t \in (0,1), \end{cases} \tag{S}$$

with the coupled multi-point boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ D_{0+}^{p_1}u(t)|_{t=1} = \displaystyle\sum_{i=1}^{N} a_i D_{0+}^{q_1}v(t)|_{t=\xi_i}, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ D_{0+}^{p_2}v(t)|_{t=1} = \displaystyle\sum_{i=1}^{M} b_i D_{0+}^{q_2}u(t)|_{t=\eta_i}, \end{cases} \tag{BC}$$

where $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \geq 3$, $p_1, p_2, q_1, q_2 \in \mathbb{R}$, $p_1 \in [1,n-2]$, $p_2 \in [1,m-2]$, $q_1 \in [0,p_2]$, $q_2 \in [0,p_1]$, ξ_i, $a_i \in \mathbb{R}$ for all $i = 1,\ldots,N$ ($N \in \mathbb{N}$), $0 < \xi_1 < \cdots < \xi_N < 1$, η_i, $b_i \in \mathbb{R}$ for all $i = 1,\ldots,M$ ($M \in \mathbb{N}$), $0 < \eta_1 < \cdots < \eta_M < 1$, and D_{0+}^{α} and D_{0+}^{β} denote the Riemann–Liouville derivatives of orders α and β, respectively.

Under sufficient conditions on the functions f and g, which can be nonsingular or singular at the points $t = 0$ and/or $t = 1$, we study the existence and multiplicity of positive solutions of problem $(S) - (BC)$. We use some theorems from the

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fixed point index theory (see [1] and [23]) and the Guo–Krasnosel'skii fixed point theorem (see [5]). By a positive solution of problem $(S)-(BC)$ we mean a pair of functions $(u,v)\in(C([0,1];\mathbb{R}_+))^2$ satisfying (S) and (BC) with $u(t)>0$ for all $t\in(0,1]$ or $v(t)>0$ for all $t\in(0,1]$. In [11], the authors give sufficient conditions for f and g such that the system (S) with $f(t,u,v)=\widetilde{f}(t,v)$ and $g(t,u,v)=\widetilde{g}(t,u)$, (denoted by $(\widetilde{S})$), and the coupled integral boundary conditions

$$\begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, & u(1)=\displaystyle\int_0^1 v(s)dH(s), \\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, & v(1)=\displaystyle\int_0^1 u(s)dK(s), \end{cases} \tag{BC$_1$}$$

has at least one or at least two positive solutions ($u(t)\ge 0$, $v(t)\ge 0$ for all $t\in[0,1]$ and $(u,v)\ne(0,0)$).

The system (S) with some positive parameters λ and μ, namely the system

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, & t\in(0,1), \\ D_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t))=0, & t\in(0,1), \end{cases} \tag{S$_1$}$$

with the boundary conditions (BC_1) has been investigated in [6] and [7] (the existence and nonexistence of positive solutions when f,g are nonnegative functions), and in [8] (the existence and multiplicity of positive solutions when f and g are sign-changing functions). The system (S_1) with $\alpha=\beta$, $\lambda=\mu$ and the coupled boundary conditions $u^{(i)}(0)=v^{(i)}(0)=0$ for $i=0,1,\dots,n-2$, $u(1)=av(\xi)$, $v(1)=bu(\eta)$ with $\xi,\eta\in(0,1)$, $0<ab\xi\eta<1$ and $n\ge 3$ was investigated in [22]. For other recent results concerning the coupled fractional boundary value problems we refer the reader to the papers [12,13,15] and [19].

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [2–4,10,14,16–18,20,21]).

The paper is organized as follows. In Section 2, we present some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations. In Section 3, we give some existence and multiplicity results for positive solutions with respect to a cone for our problem $(S)-(BC)$, where f and g are nonsingular functions. The case when f and g are singular at $t=0$ and/or $t=1$ is studied in Section 4. Finally, in Section 5, two examples are given to support our main results.

2. Auxiliary results

In this section, we present some auxiliary results that will be used to prove our main theorems.

We consider the fractional differential system

$$\begin{cases} D_{0+}^{\alpha}u(t)+x(t)=0, & t\in(0,1), \\ D_{0+}^{\beta}v(t)+y(t)=0, & t\in(0,1), \end{cases} \tag{1}$$

with the coupled multi-point boundary conditions

$$\begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, & D_{0+}^{p_1}u(t)|_{t=1}=\displaystyle\sum_{i=1}^{N}a_i D_{0+}^{q_1}v(t)|_{t=\xi_i}, \\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, & D_{0+}^{p_2}v(t)|_{t=1}=\displaystyle\sum_{i=1}^{M}b_i D_{0+}^{q_2}u(t)|_{t=\eta_i}, \end{cases} \tag{2}$$

where $\alpha\in(n-1,n]$, $\beta\in(m-1,m]$, $n,m\ge 3$, $p_1,\ p_2,\ q_1,\ q_2\in\mathbb{R}$, $p_1\in[1,n-2]$, $p_2\in[1,m-2]$, $q_1\in[0,p_2]$, $q_2\in[0,p_1]$, ξ_i, $a_i\in\mathbb{R}$ for all $i=1,\dots,N$ ($N\in\mathbb{N}$), $0<\xi_1<\cdots<\xi_N<1$, η_i, $b_i\in\mathbb{R}$ for all $i=1,\dots,M$ ($M\in\mathbb{N}$), $0<\eta_1<\cdots<\eta_M<1$, and $x,\ y:(0,1)\to\mathbb{R}$ are continuous functions.

We denote by Δ the constant

$$\Delta=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha-p_1)\Gamma(\beta-p_2)}-\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha-q_2)\Gamma(\beta-q_1)}\left(\sum_{i=1}^{N}a_i\xi_i^{\beta-q_1-1}\right)\left(\sum_{i=1}^{M}b_i\eta_i^{\alpha-q_2-1}\right).$$