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Finite-difference lattice Boltzmann model for nonlinear convection-diffusion equations

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ABSTRACT

In this paper, a finite-difference lattice Boltzmann (LB) model for nonlinear isotropic and anisotropic convection-diffusion equations is proposed. In this model, the equilibrium distribution function is delicately designed in order to recover the convection-diffusion equation exactly. Different from the standard LB model, the temporal and spatial steps in this model are decoupled such that it is convenient to study convection-diffusion problem with the non-uniform grid. In addition, it also preserves the advantage of standard LB model that the complex-valued convection-diffusion equation can be solved directly. The von Neumann stability analysis is conducted to discuss the stability region which can be used to determine the free parameters appeared in the model. To test the performance of the model, a series of numerical simulations of some classical problems, including the diffusion equation, the nonlinear heat conduction equation, the Sine-Gordon equation, the Gaussian hill problem, the Burgers–Fisher equation, and the nonlinear Schrödinger equation, have also been carried out. The results show that the present model has a second-order convergence rate in space, and generally it is also more accurate than the standard LB model.

1. Introduction

The convection-diffusion equation (CDE), as one kind of partial differential equation, has attracted considerable attention for its important role in the study of the heat and mass transfer [1]. To well understand the physical phenomena or intrinsic mechanism of the complicated dynamical system governed by the CDE, the best way is to derive its exact solution, while it is usually difficult or even impossible, especially for nonlinear CDE. With the development of computer technology, the numerical simulation, as an alternative to analytical approach, plays a significant role in solving nonlinear CDEs, and many researchers have made great efforts to develop efficient numerical approaches, such as finite-difference method [2], finite-element method [3] and finite-volume method [4].

The lattice Boltzmann (LB) method, as a mesoscopic approach, has gained a great success in the study of complex hydrodynamic problems [5], including multiphase flow [6–8], flow and mass transport in porous media [9–11], and blood

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flow [12]. On the other hand, the LB method, as a numerical solver, has also been extended to study CDEs, and some LB models for CDE were also developed in the past two decades [13–23]. For instance, Dawson et al. [13] first proposed an LB model for the isotropic CDE with a source term, and a linear equilibrium distribution function was adopted. Yu and Shi [16] also established an LB model for isotropic CDE with time delay while a quadratic equilibrium distribution was used. Shi et al. [17] also considered the problem of isotropic CDE with a source term, but mainly focused on the schemes of source term in LB model. Recently, to solve the anisotropic CDE, Yoshida and Nagaoka [23] developed a multiple-relaxation-time LB model, and also presented a discussion on the boundary condition. However, through the Chapman–Enskog analysis, we can find that the CDE can only be recovered exactly from some LB models under certain assumptions, while these assumptions may not be satisfied in practice and also influence the accuracy of LB model [24,25]. To eliminate the additional terms appeared in recovered CDE, some improved models have been developed by some researchers. For instance, Chopard et al. [24] developed a new LB model, in which a source term related to the time-derivative or spacial derivative is added in the evolution equation. Through introducing an auxiliary moment, Shi and Guo [25] proposed another LB model for nonlinear CDE, which can be used to solve isotropic, anisotropic and complex CDEs. Recently, Chai et al. [26] developed a multiple-relaxation-time LB model for general nonlinear CDE. However, the intrinsic coupling of the spatial and temporal steps is still existing in the above LB models, which brings a great limitation on the applications of the standard LB method (SLBM) [13,14,16–26].

To overcome the above limitation of the SLBM, some approaches have been developed. The first one is to adopt an interpolation method to evaluate the value of the distribution function at the non-uniformly distributed lattice points [27,28]. In this method, the collision and boundary conditions are still locally implemented on the lattice points, but compared to the SLBM, the interpolation procedure would make the algorithm more complicated, and causes unphysically numerical viscosities, as pointed out in Ref. [29]. The second one is to use the rectangular LB method [15,21,30], in which the direction-dependent weight coefficients in the equilibrium function are used to satisfy the moment conditions. Actually, van der Sman and Ernst [15] have compared the rectangular LB model with some traditional methods (finite-difference and finite-element methods), and found that it has a comparable performance with these traditional approaches. However, to recover the macroscopic CDE, some assumptions are adopted in rectangular LB model. The last one is to apply finite-difference LB method, in which some standard finite-difference schemes are used to discretize the continuous Boltzmann equation [31–34]. Compared to above two approaches, this method is much easier to implement. For this reason, in this work, we will develop a finite-difference LB model (FDLBM) for nonlinear CDEs following the idea in Ref. [31].

The rest of the paper is organized as follows. In Section 2, we present an FDLBM for nonlinear CDE, and show that through constructing the proper equilibrium distribution function and source term, the nonlinear CDE can be recovered correctly from the present model. In Section 3, the version of the FDLBM for complex nonlinear CDE is also presented. In Section 4, the stability of present FDLBM is analyzed with the von Neumann analysis [35], and is validated through performing a numerical simulation. In Section 5, a large number of simulations of some nonlinear CDEs are performed to test the present model, and finally, a brief summary is given in Section 6.

2. Finite-difference lattice Boltzmann model

The n-dimensional CDE with a source term can be written as

$$\partial_t\phi + \nabla\cdot\mathbf{B}(\phi) = \nabla\cdot[\alpha\nabla\cdot\mathbf{D}(\phi)] + F(\mathbf{x},t), \tag{1}$$

where ∇ is the spatial gradient operator, $\phi=\phi(\mathbf{x},t)$ is an unknown scaler function, $\alpha=\alpha(\mathbf{x},t)$ is the diffusion coefficient. $\mathbf{B}(\phi)$ and $\mathbf{D}(\phi)$ are the convection and diffusion terms, and $F(\mathbf{x},t)$ is the source term.

Following the idea in the previous work [25,31], the present FDLBM is developed based on the DnQb model, and the evolution equation can be written as

$$\begin{aligned} f_i(\mathbf{x},t+\Delta t) - f_i(\mathbf{x},t) + \Delta t\mathbf{c}_i\cdot\Phi = &-\frac{\Delta t\theta}{\tau}\left(f_i(\mathbf{x},t+\Delta t) - f_i^{eq}(\mathbf{x},t+\Delta t)\right) - \frac{\Delta t(1-\theta)}{\tau}\left(f_i(\mathbf{x},t) - f_i^{eq}(\mathbf{x},t)\right) \\ &+ \Delta tF_i(\mathbf{x},t) + \frac{\Delta t^2}{2}\partial_t F_i(\mathbf{x},t), \quad i=0,1,\dots,b-1, \end{aligned} \tag{2}$$

where $\{\mathbf{c}_i,\ i=0,\dots,b-1\}$ is the set of discrete velocity directions, Δt is the time step, $\Phi=\nabla f_i(\mathbf{x},t)$, τ is the dimensionless relaxation time, $\theta\in[0,1]$ is a parameter to determine whether the evolution equation is explicit or implicit. $f_i^{eq}(\mathbf{x},t)$ is the equilibrium distribution function, and $F_i(\mathbf{x},t)$ is the distribution function of the source term. Similar to the SLBM [25], the equilibrium distribution function is still defined as

$$f_i^{eq} = \omega_i\left[\phi + \frac{\mathbf{c}_i\cdot\mathbf{B}(\phi)}{c_s^2} + \frac{(\mathbf{C}(\phi) - c_s^2\phi\mathbf{I}) : (\mathbf{c}_i\mathbf{c}_i - c_s^2\mathbf{I})}{2c_s^4}\right], \tag{3}$$

where ω_i is the weight coefficient, c_s is the sound speed related to lattice velocity, and $\mathbf{I}$ is the unit tensor. $\mathbf{C}(\phi)=\mathbf{C}_0(\phi)+c_s^2\mathbf{D}(\phi)$ is the second-order moment of f_i^{eq}, and $\mathbf{C}_0(\phi)$ is an auxiliary-moment, which can be used to remove some additional terms in the recovered CDE.

Here we only take the D2Q9 lattice model as an example, and present a detailed Chapman–Enskog analysis on how to derive the CDE from the present model. In the D2Q9 lattice model, $\omega_0=4/9,\ \omega_{1\sim4}=1/9,\ \omega_{5\sim8}=1/36,\ \{\mathbf{c}_i,\ i=0,\dots,8\}=$