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Short Communication

Principal minor version of Matrix-Tree theorem for mixed graphs

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A R T I C L E I N F O

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A B S T R A C T

In Yu, et al. (2017), an analytical expression of the determinant of the Hermitian (quasi-)Laplacian matrix of mixed graphs has been proven. In this paper, we are going to extend those results and derive an analytical expression for the principal minors of the Hermitian (quasi-)Laplacian matrix, which is the principal minor version of the Matrix-Tree theorem.

1. Introduction

Let $G=(V(G),E(G))$ be a simple undirected of Mechatronics and Biomedical Computer Science graph with vertex set $V(G)$ and edge set $E(G)$. A *mixed graph* M is obtained from an undirected graph G by orienting a subset of its edges. We call G the *underlying graph* of M, denoted by M_u. The vertex set of M is denoted by $V(M)$, which equals $V(G)$. The edge set $E(M)$ is the union of the set of undirected edges $E_0(M)$ and the set of directed edges (arcs) $E_1(M)$. We distinguish an undirected edge by $x \leftrightarrow y$, while the directed edge (arcs) is to be $x \to y$, if the orientation is from x to y. If we do not consider any direction, we just write xy is an edge of M.

Similar to the adjacency matrix of undirected graphs, Liu and Li [6] and Guo and Mohar [4] independently introduced a definition for the adjacency matrix of a mixed graphs as follows. The *Hermitian adjacency matrix* of a mixed graph M of order n is an $n \times n$ matrix $H(M)=(h_{kl})$, where $h_{kl}=-h_{lk}=i$ $(i=\sqrt{-1})$, if there exists an orientation from v_k to v_l and $h_{kl}=h_{lk}=1$, if there exists an edge between v_k and v_l without any orientation; otherwise $h_{kl}=0$.

The *Hermitian Laplacian matrix* $L(M)$ of a mixed graph M has been introduced in [9]. It's defined by $D(M)-H(M)$ where

$$D(M)=\mathrm{diag}(d_1,d_2,\ldots,d_n)$$

is a diagonal matrix and d_i is the degree of the vertex v_i in the underlying graph M_u. The matrix $Q(M)=D(M)+H(M)$ is called the *Hermitian quasi-Laplacian matrix* of M, which has been introduced in [10]. Obviously these two matrices are Hermitian and all eigenvalues are real. Let $S(M)=(s_{ke})$ be a $n \times m$ matrix indexed by the vertex and the edge of the mixed

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graph M, where s_{ke} is a complex number and $|s_{ke}| = 1$ or 0. Moreover,

$$s_{ke} = \begin{cases} -s_{le}, & \text{if } v_k \leftrightarrow v_l \\ -i \cdot s_{le}, & \text{if } v_k \rightarrow v_l \\ i \cdot s_{le}, & \text{if } v_k \leftarrow v_l \\ 0, & \text{otherwise.} \end{cases}$$

It is evident that $S(M)$ is not unique. A matrix $S(M)$ is referred to as *incidence matrix* of M. Moreover, $L(M) = S(M)S^*(M)$ and $L(M)$ is positive semi-definite. A mixed graph is called *Laplacian singular* if its Hermitian Laplacian matrix is singular. Otherwise, it is *Laplacian non-singular*. Up to now, there has been only little research activity towards investigating the Hermitian spectra of a mixed graph (see [2,4,6,8–10]).

A $i_1 - i_k$-walk W in a mixed M is a sequence $W : v_{i_1} v_{i_2} \cdots v_{i_k}$ of vertices such that for $1 \le s \le k-1$ we have $v_{i_s} \leftrightarrow v_{i_{s+1}}$ or $v_{i_s} \rightarrow v_{i_{s+1}}$ or $v_{i_s} \leftarrow v_{i_{s+1}}$. A $i_1 - i_k$-walk W is called *even* (*odd*) if k is even (odd). The value of a mixed walk $W = v_1 v_2 v_3 \cdots v_l$ is $h(W) = h_{12} h_{23} \cdots h_{(l-1)l}$. A mixed walk is *positive* or *negative* if $h(W) = 1$ or $h(W) = -1$, respectively. A mixed walk is called *imaginary* if $h(W) = \pm i$. Note that for one direction the value of a mixed walk or a mixed cycle is α; for the reverse direction its value is $\overline{\alpha}$. Thus, if the value of a mixed cycle is 1 (resp. -1) for one direction, then its value is 1 (resp. -1) for the reverse direction. In such situations, we just call this mixed cycle a positive (negative) mixed cycle without mentioning any direction. A graph is *positive* (resp. *negative*) if each of it's mixed cycles is positive (resp. negative). An induced subgraph of M is an induced subgraph of its underlying graph G with the same orientation. For a subgraph H of M, let $M - H$ be the subgraph obtained from M by deleting all vertices of H and all incident edges. For $V_1 \subseteq V(M)$, $M - V_1$ is the subgraph obtained from M by deleting all vertices in V_1 and all their incident edges.

Bapat [1] introduced a real Laplacian matrix on mixed graph and investigated the Matrix-Tree Theorem based on this matrix. Inspired by Bapat et al. [1], we here study the same question in terms of the Hermitian (quasi-)Laplacian matrix.

2. Hermitian Laplacian matrix of mixed graphs

To prove the main result, we apply the definition of a substructure of a graph. This definition can be found in [1,3]. Let M be a mixed graph. A substructure R of M is simply a pair (V_R, E_R) where $V_R \subseteq V(M)$, $E_R \subseteq E(M)$. Suppose we have given a substructure; then, we may delete some of the vertices but retain the edges incident to those vertices and their orientation; equally we could delete some edges and their orientations, or possibly both. In this case, however, we will retain at least one end-vertex of every edge. In particular, if we take a spanning tree of a connected mixed graph on n vertices and delete one vertex (but not the edges incident with it), the resulting substructure which represents a *rootless tree* has $n-1$ vertices and $n-1$ edges. Note that a rootless tree will be disconnected if the degree of the deleted vertex was larger than one. Each substructure R of a mixed graph M gives rise to a submatrix $S(R)$ of the incident matrix $S(M)$ with $|V_R|$ rows corresponding to V_R and $|E_R|$ columns corresponding to E_R. This submatrix $S(R)$ is called the incidence matrix of the substructure R. The following result is related to the determinant of the incidence matrix of a rootless tree and play a crucial role in our proof of the main result.

Lemma 1. *Let M be a mixed graph with an equal number of vertices and edges and v be a pendant vertex of M. Let $S(M) = (s_{ij})$ be the incidence matrix of M. Then*

$$|\det S(M)| = |\det S(M')|,$$

where M' is a mixed graph obtained from M by deleting v with its incident edge.

Proof. Let e be the edge incident to v. After relabeling the vertices (if necessary), the first row and the first column of the incidence matrix $S(M)$ correspond to the vertex v and its incident e. By expanding the first row, we have

$$\det S(M) = s_{ve} \cdot \det S(M'),$$

where s_{ve} is the entry of $S(M)$ located in the first row and first column. Note that $|s_{ve}| = 1$. So we have $|\det S(M)| = |\det S(M')|$. □

The following statements follow straightforwardly by using Lemma 1.

Lemma 2. *Let R be a rootless tree. Then $|\det S(R)| = 1$.*

Lemma 3. *[10] Let C be a mixed cycle. Then $\det L(C) = 2 - [h(C) + \overline{h(C)}]$.*

Lemma 4. *[10] Let M be a connected mixed unicyclic graph with cycle C. Then $\det L(M) = \det L(C) = 2 - [h(C) + \overline{h(C)}]$.*

Let $Q_{k,m}$ be the set of strictly increasing sequences of k $(1 \le k \le m)$ integers chosen from $\{1, 2, \ldots, m\}$ and $Q_{r,n}$ be the set of strictly increasing sequences of r $(1 \le r \le n)$ integers chosen from $\{1, 2, \ldots, n\}$. Assume that A is a $m \times n$ matrix and k, r are positive integers satisfying $1 \le k \le m$, $1 \le r \le n$.

For $\alpha = (i_1, i_2, \ldots, i_k) \in Q_{k,m}$ and $\beta = (j_1, j_2, \ldots, j_r) \in Q_{r,n}$, $A[\alpha|\beta]$ is a $k \times r$ submatrix of A, whose k rows correspond to α and r columns correspond to β. $A(\alpha|\beta)$ is the $(m-k) \times (n-r)$ submatrix of A obtained by deleting the rows indexed by α; the columns are indexed by β.