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On the local convergence of a Newton–Kurchatov-type method for non-differentiable operators☆

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ABSTRACT

By means of a nice idea, a Newton–Kurchatov type iterative process is constructed for solving nonlinear equations in Banach spaces. We analyze the local convergence of this iterative process. This study have an important and novel feature, since it is applicable to non-differentiable operators. So far, most of the local convergence results considered by other authors may apply only to differentiable operators due to the conditions that are required on the solution of the nonlinear equation.

1. Introduction

A lot of problems from computational sciences and other disciplines can be brought, using Mathematical Modeling [2], in the form of the equation

$$H(x) = 0. \tag{1}$$

To give sufficient generality to the problem of approximating a solution of a nonlinear equation by iterative methods, throughout the paper we consider that H: $\Omega \subseteq X \to Y$ is a nonlinear operator defined on a nonempty open convex subset Ω of a Banach space X with values in a Banach space Y, so that many scientific and engineering problems can be written as a nonlinear equation in Banach space; for example, nonlinear integral equations, initial value problems, matrix equations, nonlinear PVF, etc..

In general, the roots of nonlinear equation (1) cannot be expressed in a closed form and this problem is commonly carried out applying iterative methods. If H is a differentiable operator, Newton's method [3,11] is the most used iteration to solve (1), due to its computational efficiency, and it is given by

$$x_{n+1} = x_n - \left[H'(x_n)\right]^{-1} H(x_n), \quad n \geq 0; \quad x_0 \in \Omega \text{ is given}, \tag{2}$$

but this method needs the existence of H'. For this reason, Newton's method cannot be applied when H is non-differentiable. In this situation, as it is known, it is necessary to approximate the operator H'. A commonly applied approximation is the use of divided differences. We shall use, as in [14], the known definition for the divided differences of an operator. First, we denote the space of bounded linear operators from X to Y by $\mathcal{L}(X, Y)$. Second, an operator $[x, y; D] \in \mathcal{L}(X, Y)$ is called a first order divided difference for the operator D: $\Omega \subseteq X \to Y$ on the points x and y $(x \neq y)$ if the following equality holds

$$[x, y; D](x - y) = D(x) - D(y). \tag{3}$$

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Using this definition, in [13], Kurchatov proposed a linear interpolation method which is as simple as Newton's method, has the same rate of convergence but, in contrast, it do not use H'. The iterative method proposed by Kurchatov is defined by the following algorithm:

$$x_{n+1} = x_n - [x_{n-1}, 2x_n - x_{n-1}; H]^{-1} H(x_n), \quad n \geq 0; \quad x_{-1}, x_0 \in \Omega \text{ are given.} \tag{4}$$

In this paper, we consider the case in which the operator H is continuous but nondifferentiable, such that

$$H(x) = F(x) + G(x),$$

where F, G: $\Omega \subseteq X \to$, Y are nonlinear operators, F is differentiable and G is continuous but nondifferentiable. Then, for approximating a root of Eq. (1), we apply the following Newton–Kurchatov-type iterative process:

$$\begin{cases} x_{-1}, \; x_0 \in \Omega \text{ are given,} \\ x_{n+1} = x_n - \left(F'(x_n) + [x_{n-1}, 2x_n - x_{n-1}; G]\right)^{-1} H(x_n), \quad n \geq 0. \end{cases} \tag{5}$$

Notice that if H is differentiable ($G = 0$), Newton's method (2) is obtained and, if we consider $F = 0$, we then obtain Kurchatov's method (4).

There are two advantages of process (5): first, the differentiable part of the operator is considered in the optimal situation, namely $F'(x_n)$; and second, for the nondifferentiable part, iteration (4) is considered, which improves the results given by the Secant method [1] because (4) has quadratic convergence.

An important aspect that has been taken into account when choosing an iterative method to approximate a solution of an equation is the accessibility of the iterative process, which shows the domain of starting points from which the iterative process converges to a solution of the equation. The location of starting approximations, from which the iterative methods converge to a solution of the equation, is a difficult problem to solve. This location is from the study of the convergence that is made of the iterative process considered. To analyze the accessibility of an iterative process two types of studies can be done, when we are interested in proving the convergence of iterative processes: local and semilocal. In this paper, we focus our attention on the analysis of the local convergence of sequence (5). The local study of the convergence is based on demanding conditions to a solution x^*, from certain conditions on the operator H, and provide the so-called ball of convergence of iterative process, that shows the accessibility to x^* from the initial approximations belonging to the ball.

Occasionally, the study of the local convergence of derivative-free iterative processes shows a small contradiction. There are many known results of local convergence (see [4,5,10,12,16,18] and references therein given) which usually include the condition of the existence of the operator $[H'(x^*)]^{-1}$, forcing the operator H to be differentiable. However, in this paper, we obtain a new type of result for the local convergence from requiring a new type of assumptions. It is also interesting to note that we can deduce a result of local convergence for non-differential operators. Moreover, from this new result of local convergence, we obtain some results of local convergence for the Newton and Kurchatov methods for some particular cases of the operator H.

This paper is organized as follows. In Section 2, we obtain a local convergence result for the Newton–Kurchatov-type iterative process given in (5) when non-differentiable operators are considered, obtaining a ball of convergence. In Section 3, relaxing the convergence conditions considered previously, an improvement of the result of local convergence given in the previous section is obtained. Finally, in Section 4, considering particular cases of the iterative process (5), and taking into account the study previously realized, we present some local convergence results for Newton and Kurchatov methods.

Throughout the paper we denote $\overline{B(x, \varrho)} = \{y \in X; \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in X; \|y - x\| < \varrho\}$.

2. Local convergence

The local convergence results for iterative method (5) require conditions on the operators F, G and the solution x^* of Eq. (1). Note that a local result provides what we call ball of convergence, denoted by $B(x^*, r)$. From the value r, the ball of convergence gives information about the accessibility of the solution x^*. In this section, we analyze the local convergence of iterative process (5).

First, we consider the following generic conditions on the operators F and G:

(I) For $x, y \in \Omega$,

$$\|F'(x) - F'(y)\| \leq \omega_1(\|x - y\|) \tag{6}$$

where $\omega_1 : \mathbb{R}_+ \to \mathbb{R}_+$, with $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$, is a continuous nondecreasing function and assume that there exists a continuous and nondecreasing function $h : [0, 1] \to \mathbb{R}_+$, such that $\omega_1(tz) \leq h(t)\omega_1(z)$, with $t \in [0, 1]$ and $z \in [0, \infty)$. In addition, we denote $T = \int_0^1 h(t)\, dt$.

(II) We suppose that there exists $[z, w; G]$ for each pair of distinct points $z, w \in \Omega$ and

$$\|[x, u; G] - [y, v; G]\| \leq \omega_2(\|x - y\|, \|u - v\|); \quad x, y, u, v \in \Omega, \tag{7}$$

where $\omega_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function in its two arguments.