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Computing the permanental polynomials of graphs

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ABSTRACT

Let M be an $n \times n$ matrix with entries m_{ij} $(i, j = 1, 2, \dots, n)$. The *permanent* of M is defined to be

$$\mathrm{per}(M) = \sum_{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)},$$

where the sum is taken over all permutations σ of $\{1, 2, \dots, n\}$. The *permanental polynomial* of M is defined by $\mathrm{per}(xI_n - M)$, where I_n is the identity matrix of size n. In this paper, we give recursive formulas for computing permanental polynomials of the Laplacian matrix and the signless Laplacian matrix of a graph, respectively.

1. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G, denoted by $A(G) = (a_{ij})_{n\times n}$, is an $n \times n$ symmetric matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent and 0 otherwise. Let $d_i = d(v_i) = d_G(v_i)$ be the degree of vertex v_i in G and $D(G) = \mathrm{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* and *signless Laplacian matrix* of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively.

Given an $n \times n$ matrix M with entries m_{ij} $(i, j = 1, 2, \dots, n)$, the *permanent* of M is defined to be

$$\mathrm{per}(M) = \sum_{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)},$$

where the sum is taken over all permutations σ of $\{1, 2, \dots, n\}$. Denote by

$$\psi(M; x) = \mathrm{per}(xI_n - M),$$

or simply $\psi(M)$, the *permanental polynomial* of M, where I_n is the identity matrix of size n. In particular, we call $\psi(A(G))$ (respectively, $\psi(L(G))$, $\psi(Q(G))$) the *adjacency* (respectively, *Laplacian, signless Laplacian*) *permanental polynomial* of a graph G.

It is known [33] that computing the permanent of a square matrix is a #P-complete problem even for (0, 1)-matrix. Thus, it is difficult to compute the permanental polynomials of graphs. The adjacency permanental polynomials of graphs were

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first studied by Turner [31], and subsequently by Merris et al. [27] and Kasum et al. [20]. Since then, a lot of works were done on the adjacency permanental polynomials of graphs, including:

1. Characterizing relations between the adjacency permanental and characteristic polynomials of graphs [2,10,11,17,27,36,37];
2. Computing the adjacency permanental polynomials of some chemical graphs [6–8,19,21,24,32];
3. Characterizing which graphs are determined by their adjacency permanental polynomials [25–27,34,35,38]; etc..

The Laplacian permanental polynomials of graphs were first studied by Merris et al. [27] and Merris [28], and then Brualdi and Goldwasser [4], Goldwasser [16] and Bapat [3] studied the permanents of the Laplacian matrices of graphs (note that $\mathrm{per}(L(G)) = (-1)^{|V(G)|}\psi(L(G);0)$ for a graph G). Recently, new progress on the Laplacian permanental polynomials of graphs, in particular, the permanents of the Laplacian matrices of graphs has been made in [9,14,15]. Meanwhile, there have been some work on the permanents of the signless Laplacian matrices of graphs [22,23]. For more information on the permanents of matrices, we refer the reader to the monograph [29].

Let G be a graph with a vertex subset $S \subseteq V(G)$. Denote by $G-S$ the graph obtained by deleting the vertices S from G together with all edges incident with S. In particular, if $S=\{v\}$ with $v \in V(G)$, we write $G-\{v\}$ simply by $G-v$. In [1], Borowiecki and Jóźwiak gave the recursive formulas to calculate the adjacency permanental polynomials of graphs, which are stated as follows.

Theorem 1.1.

(a) *Let v be a vertex of G, $\mathscr{C}_G(v)$ the set of cycles of G containing v and $N(v)$ the set of vertices of G adjacent to v. Then*

$$\psi(A(G);x) = x\psi(A(G-v)) + \sum_{u\in N(v)} \psi(A(G-\{u,v\})) + 2\sum_{C\in\mathscr{C}_G(v)} (-1)^{|V(C)|}\psi(A(G-V(C))).$$

(b) *Let $e=uv$ be an edge of G and $\mathscr{C}_G(e)$ the set of cycles containing e in G. Denote by $G-e$ the graph obtained by deleting the edge e from G. Then*

$$\psi(A(G);x) = \psi(A(G-e)) + \psi(A(G-\{u,v\})) + 2\sum_{C\in\mathscr{C}_G(e)} (-1)^{|V(C)|}\psi(A(G-V(C))).$$

The main purpose of this paper is to give the recursive formulas to compute the Laplacian and the signless Laplacian permanental polynomials of graphs, respectively. Our main results are stated in the following, where $L_S(G)$ (respectively, $Q_S(G)$) denotes the principal submatrix of $L(G)$ (respectively, $Q(G)$) formed by deleting the row and column corresponding to all vertices of $S \subseteq V(G)$, and $L_e(G)$ (respectively, $Q_e(G)$) denotes the principal submatrix of $L(G)$ (respectively, $Q(G)$) formed by deleting the row and column corresponding to the vertices of the edge $e \in E(G)$. In particular, if $S=\{v\}$ with $v \in V(G)$, then $L_{\{v\}}(G)$ and $Q_{\{v\}}(G)$ are simply written as $L_v(G)$ and $Q_v(G)$, respectively.

Theorem 1.2.

(a) *Let v be a vertex of a graph G, $\mathscr{C}_G(v)$ the set of cycles of G containing v and $N(v)$ the set of vertices of G adjacent to v. Then*

$$\psi(L(G);x) = (x-d(v))\psi(L_v(G)) + \sum_{u\in N(v)} \psi(L_{vu}(G)) + 2\sum_{C\in\mathscr{C}_G(v)} \psi(L_{V(C)}(G)).$$

(b) *Let $e=uv$ be an edge of a graph G. Denote by $G-e$ the graph obtained by deleting the edge e from G. Then*

$$\psi(L(G);x) = \psi(L(G-e)) - \psi(L_v(G-e)) - \psi(L_u(G-e)) + 2\psi(L_{uv}(G)) + 2\sum_{C\in\mathscr{C}_G(e)} \psi(L_{V(C)}(G)).$$

Theorem 1.3.

(a) *Let v be a vertex of a graph G, $\mathscr{C}_G(v)$ the set of cycles of G containing v and $N(v)$ the set of vertices of G adjacent to v. Then*

$$\psi(Q(G);x) = (x-d(v))\psi(Q_v(G)) + \sum_{u\in N(v)} \psi(Q_{vu}(G)) + 2\sum_{C\in\mathscr{C}_G(v)} (-1)^{|V(C)|}\psi(Q_{V(C)}(G)).$$

(b) *Let $e=uv$ be an edge of G. Denote by $G-e$ the graph obtained by deleting the edge e from G. Then*

$$\psi(Q(G);x) = \psi(Q(G-e)) - \psi(Q_v(G-e)) - \psi(Q_u(G-e)) + 2\psi(Q_{uv}(G)) + 2\sum_{C\in\mathscr{C}_G(e)} (-1)^{|V(C)|}\psi(Q_{V(C)}(G)).$$

The paper is organized as follows. In Section 2, we first prove Theorem 1.2, and then we give recursive formulas to compute the Laplacian permanental polynomials of some graphs with specific structures. In Section 3, we prove Theorem 1.3, and we also give recursive formulas to compute the signless Laplacian permanental polynomials of some graphs with specific structures. Applications of Theorems 1.2 and 1.3 are given in Section 4.