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Nonoscillation theorems for second-order linear difference equations via the Riccati-type transformation, II

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ABSTRACT

The present paper deals with nonoscillation problem for the second-order linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \qquad n = 1, 2, \ldots,$$

where $\{b_n\}$ and $\{c_n\}$ are positive sequences. All nontrivial solutions of this equation are nonoscillatory if and only if the Riccati-type difference equation

$$q_n z_n + \frac{1}{z_{n-1}} = 1$$

has an eventually positive solution, where $q_n = c_n^2/(b_n b_{n+1})$. Our nonoscillation theorems are proved by using this equivalence relation. In particular, it is focusing on the relation of the triple $(q_{3k-2}, q_{3k-1}, q_{3k})$ for each $k \in \mathbb{N}$. Our results can also be applied to not only the case that $\{b_n\}$ and $\{c_n\}$ are periodic but also the case that $\{b_n\}$ or $\{c_n\}$ is non-periodic. To compare the obtained results with previous works, we give some concrete examples and those simulations.

1. Introduction

The Riccati transformation is a very important tool for studying nonoscillation problem of second-order linear difference equations as well as ordinary differential equations. It is known that there are several types of Riccati transformations. For example, Hooker et al. [15], Hooker and Patula [16] and Kwong et al. [19] have presented three kinds of Riccati transformations for the second-order linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \qquad n = 1, 2, \ldots, \tag{1.1}$$

where $\{b_n\}$ and $\{c_n\}$ are sequences satisfying $b_n > 0$ for $n \in \mathbb{N}$ and $c_n > 0$ for $n \in \mathbb{N} \cup \{0\}$, respectively. Those Riccati transformations are expressed by $w_n = x_{n+1}/x_n$, $y_n = c_n x_{n+1}/x_n$ and $z_n = b_{n+1} x_{n+1}/(c_n x_n)$. Here, we assume that there exists an $M \in \mathbb{N}$ such that $x_n > 0$ for $n \geq M$. The transformations lead to the first-order non-linear difference equations

$$c_n w_n + \frac{c_{n-1}}{w_{n-1}} = b_n,$$

$$y_n + \frac{c_{n-1}^2}{y_{n-1}} = b_n$$

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and

$$q_n z_n + \frac{1}{z_{n-1}} = 1, \quad q_n = \frac{c_n^2}{b_n b_{n+1}} \tag{1.2}$$

with $n = M+1, M+2, \dots,$ respectively (see also the books [1, Chapter 6], [9, Chapter 7]) . Although the transformation

$$z_n = \frac{b_{n+1}x_{n+1}}{c_n x_n}$$

is the most complicated one out of those three, Eq. (1.2) is easiest to use because the coefficient of (1.2) is only one.

It is clear that Eq. (1.1) has the trivial solution $\{x_n\}$; that is, $x_n = 0$ for $n \ge 0$. The others are called non-trivial solutions. A non-trivial solution of (1.1) is said to be *oscillatory* if, for every $N \in \mathbb{N}$ there exists an $n \ge N$ such that $x_n x_{n+1} \le 0$. Otherwise, it is said to be *nonoscillatory*. Hence, a nonoscillatory solution $\{x_n\}$ of (1.1) satisfies that $x_n > 0$ for n sufficiently large or $x_n < 0$ for n sufficiently large. Since Eq. (1.1) is linear, $\{x_n\}$ is a solution of (1.1) if and only if $\{-x_n\}$ is also a solution of (1.1). Hence, it is sufficient to consider that a nonoscillatory solution $\{x_n\}$ of (1.1) continues being positive for all large n.

As known well, Sturm's separation theorem holds for Eq. (1.1). About the proof of Sturm's separation theorem concerning linear difference equations, see [9, pp. 321–322] for example. From Sturm's separation theorem it follows that if one non-trivial solution of (1.1) is nonoscillatory, then all its non-trivial solutions are nonoscillatory. Hence, oscillatory solutions and nonoscillatory solutions do not coexist in Eq. (1.1).

Using Eq. (1.2) equivalent to (1.1), Hooker et al. [15] have proved the following results.

Theorem A. *If $q_n \ge 1/(4-\varepsilon)$ for some $\varepsilon > 0$ and for all sufficiently large n, then all non-trivial solutions of (1.1) are oscillatory.*

Theorem B. *If $q_n \le 1/4$ for all sufficiently large n, then all non-trivial solutions of (1.1) are nonoscillatory.*

As can be seen from Theorems A and B, the constant 1/4 is a critical value that divides oscillation and nonoscillation of solutions of (1.1). Such a value is called an *oscillation constant*. It seems to be appropriate that the constant 1/4 appears in Theorems A and B, because it often becomes the oscillation constant for some ordinary differential equations. For example, it is well-known that all non-trivial solutions of the Euler differential equation

$$x'' + \frac{\gamma}{t^2}x = 0$$

are nonoscillatory if and only if $\gamma \le 1/4$ (for example, see [14,18,21,26]). In this sense, it is not exaggeration even if we say that Theorems A and B have similarity between the results of ordinary differential equations. After that, Hooker et al. [16] and Kwong et al. [19] improved the sufficient condition was given in Theorem A which guarantees that all nontrivial solutions of (1.1) are oscillatory.

Equation (1.1) can be rewritten as the self-adjoint difference equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + p_n x_n = 0, \tag{1.3}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$ and

$$p_n = c_{n-1} + c_n - b_n$$

for $n \in \mathbb{N}$. The oscillation and nonoscillation of (1.3) and more generalized equations have been considered extensively by many authors. For example, see [1–5,9,12,17] and the references cited therein. Chen and Erbe [4] discussed the oscillation and nonoscillation properties of (1.3) and obtained oscillation and nonoscillation criteria by using Riccati techniques. Their main assumptions were

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} p_j > -\infty \tag{1.4}$$

and others. Since the beginning of this century, oscillation and nonoscillation criteria are now being actively reported for the self-adjoint difference equation

$$\Delta(c_{n-1}\Phi(\Delta x_{n-1})) + p_n\Phi(x_n) = 0, \tag{1.5}$$

which is a generalization of (1.3). Here, $\Phi(z)$ is a real-valued nonlinear function defined by

$$\Phi(z) = \begin{cases} |z|^{p-2}z & \text{if } z \ne 0, \\ 0 & \text{if } z = 0 \end{cases}$$

for $z \in \mathbb{R}$ with $p > 1$ a fixed real number. For example, see [7,10,11,13,20,22–24,27]. Eq. (1.5) is often called a *half-linear* difference equation. Most of these results emphasized similarity of difference equations (1.3) and (1.5) and the differential equation

$$(c(t)x')' + p(t)x = 0$$

and its generalization

$$(c(t)\Phi(x'))' + p(t)\Phi(x) = 0,$$