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Polynomiography for the polynomial infinity norm via Kalantari's formula and nonstandard iterations

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ABSTRACT

In this paper, an iteration process, referred to in short as MMP, will be considered. This iteration is related to finding the maximum modulus of a complex polynomial over a unit disc on the complex plane creating intriguing images. Kalantari calls these images polynomiographs independently from whether they are generated by the root finding or maximum modulus finding process applied to any polynomial. We show that the images can be easily modified using different MMP methods (pseudo-Newton, MMP-Householder, methods from the MMP-Basic, MMP-Parametric Basic or MMP-Euler–Schröder Families of Iterations) with various kinds of non-standard iterations. Such images are interesting from three points of views: scientific, educational and artistic. We present the results of experiments showing automatically generated non-trivial images obtained for different modifications of root finding MMP-methods. The colouring by iteration reveals the dynamic behaviour of the used root finding process and its speed of convergence. The results of the present paper extend Kalantari's recent results in finding the maximum modulus of a complex polynomial based on Newton's process with the Picard iteration to other MMP-processes with various non-standard iterations.

1. Introduction

Kalantari defined polynomiography as the art and science of visualisation in approximation of the zeros of complex polynomials via fractal and non-fractal images created using the mathematical convergence properties of iteration functions [1,2]. The well-known Newton method, as well as methods from the Basic Family and Euler–Schröder Family of Iterations will be used as iteration functions. The polynomiograph is a single two-dimensional image that presents the visualisation process of root finding for a given polynomial. Polynomiography, as a method of producing interesting graphics that could be widely used, was patented by Kalantari in the USA in 2005 [1].

In [3,4], the authors presented a survey of some modifications of Kalantari's polynomiography based on the classic Newton's and the higher order Newton-like root finding methods for complex polynomials. Instead of the standard Picard's iteration, several different iteration processes were used. By combining different kinds of iterations, different convergence tests and different colouring methods they obtained a great variety of polynomiographs [4].

Recently, Kalantari presented the Maximum Modulus Principle (MMP) for complex polynomials and related it to the pseudo-Newton method together with some illustrative examples of polynomiographs [5]. The pseudo-Newton method

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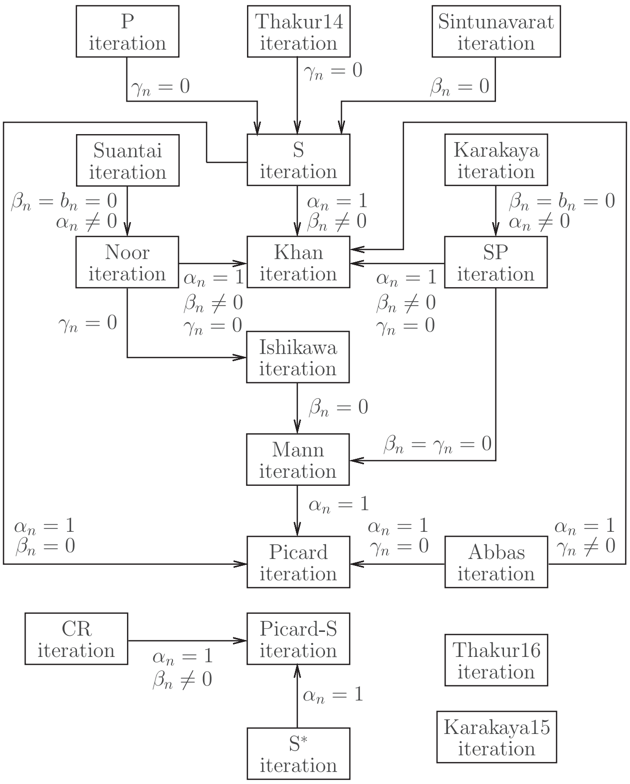

Fig. 1. The diagram of iterations' dependencies.

produces intriguing images different from those obtained via the classic Newton's root finding process. In this paper, following [5], we explore further modifications of the algorithms for polynomiograph rendering obtained with the help of various iterations and root finding methods in their pseudo versions, which we call MMP methods. In comparison to [3,4], we extend the list of iterations adding new iterations that have been presented in the literature recently. The actual list of iterations contains 18 items. Dependencies between iterations have been investigated and are presented on the diagram in Fig. 1.

The paper is organised as follows. Section 2 presents the Maximum Modulus Principle for polynomials and its connection with the pseudo-Newton method. Section 3 gives the definitions of the 18 types of iterations used in the fixed point theory and known from the literature. The following section, Section 4, describes selected root finding methods for a specific pseudo-polynomial. Section 5 is devoted to some modifications of the methods presented in Section 4. These modifications can be easily obtained using one of the non-standard iterations, instead of the Picard iteration. Section 6 describes the polynomiograph generation algorithm and Section 7 shows examples of polynomiographs. Finally, Section 8 concludes the paper and shows future directions on this subject.

2. Maximum Modulus Principle for polynomials

Denote by p any non-constant complex polynomial on domain $D = \{z \in \mathbb{C} : |z| \le 1\}$. Next, state the following Maximum Modulus Problem: find a local maximum of $|p(z)|$ on D. It is known that, in this case, the Maximum Modulus Principle is satisfied [6] and states that

$$\|p\|_\infty = \max\{|p(z)| : z \in D\} \tag{1}$$

is attained at a boundary point of D. Further, a point $z_* \in D$ is a local maximum of $|p(z)|$ over D if and only if [5]

$$z_* = \left(\frac{p(z_*)}{p'(z_*)}\right) \Big/ \left(\left|\frac{p(z_*)}{p'(z_*)}\right|\right). \tag{2}$$

Formula (2) can be used to test if a given z is a local maximum of $|p(z)|$ on D. From (2) it follows that if z_* is a local maximum of $|p(z)|$ over D, then z_* is a fixed point of

$$F(z) = \left(\frac{p(z)}{p'(z)}\right) \Big/ \left(\left|\frac{p(z)}{p'(z)}\right|\right). \tag{3}$$