Applied Mathematics and Computation 307 (2017) 239–243

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Enumeration of spanning trees of middle graphs

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ARTICLE INFO

Keywords:
Line graph
Spanning tree
Middle graph
Generalized Wye-Delta transform

ABSTRACT

Let G be a simple graph with n vertices and m edges, and Δ and δ the maximum degree and minimum degree of G. Suppose G' is the graph obtained from G by attaching $\Delta - d_G(v)$ pendent edges to each vertex v of G. Huang and Li (Bull. Aust. Math. Soc. 91(2015), 353–367) proved that if G is regular (i.e., $\Delta = \delta, G = G'$), then the middle graph of G, denoted by $M(G)$, has $2^{m-n+1}\Delta^{m-1}t(G)$ spanning trees, where $t(G)$ is the number of spanning trees of G. In this paper, we prove that $t(M(G))$ can be expressed in terms of the summation of weights of spanning trees of G with some weights on its edges. Particularly, we prove that if G is irregular (i.e., $\Delta \neq \delta$), then $t(M(G')) = 2^{m-n+1}\Delta^{m+k-1}t(G)$, where k is the number of vertices of degree one in G'.

1. Introduction

The graphs considered in this paper are simple, if not specified. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denote the maximum degree and minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively. We define the revision of G, denoted by G', as the graph obtained from G by attaching $\Delta(G) - d_G(v)$ pendent edges to each vertex v of G, where $d_G(v)$ is the degree of vertex v of G. Obviously, $G' = G$ if G is regular and G' has $n + \sum_{v\in V(G)}[\Delta(G) - d_G(v)] = n + n\Delta(G) - 2m$ vertices, where n vertices have degree $\Delta(G)$ and $n\Delta(G) - 2m$ vertices have degree one. For the graph G illustrated in Fig. 1(a), G' is illustrated in Fig. 1(b). Suppose $G = (V(G), E(G))$ is a connected graph with vertex set $V(G)$ and edge set $E(G)$. Given an edge $e = (u, v)$ of G, let $V(e)$ denote the set consisting of the two end vertices of e. Now we can define three related graphs – the line graph $L(G)$, the subdivision graph $S(G)$, and the middle graph $M(G)$ of G as follows (cf. e.g., [3]):

Line graph: The vertices of $L(G)$ are the edges of G. Two edges of G that share a common vertex are considered to be adjacent in $L(G)$.

Subdivision graph: $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.

Middle graph: $M(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

For the graph G in Fig. 1(a), the corresponding subdivision graph $S(G)$, line graph $L(G)$, and the middle graph $M(G)$ are illustrated in Fig. 2(a)–(c), respectively. Given $G = (V(G), E(G))$, where $E(G) \subset \binom{V(G)}{2}$, we define two other sets that we use frequently:

$$EE(G) := \{(e, f) : e, f \in E(G), |V(e) \cap V(f)| = 1\},$$

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[1] This work is supported by NSC Grant (11571139).

http://dx.doi.org/10.1016/j.amc.2017.02.040

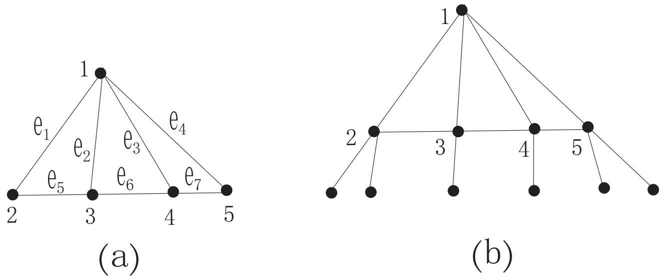

Fig. 1. (a) A connected graph G. (b) The graph G' of G.

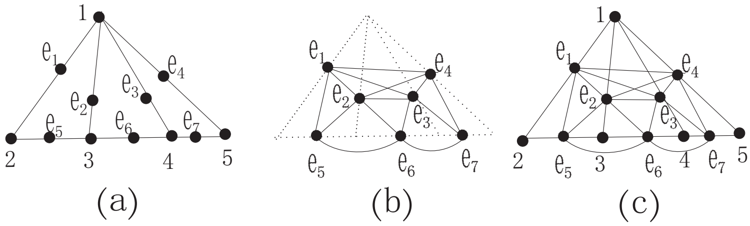

Fig. 2. (a) The subdivision graph $S(G)$ of the graph G in Fig. 1(a). (b) The line graph $L(G)$ of the graph G in Fig. 1(a), which is illustrated with solid lines. (c) The middle graph $M(G)$ of the graph G in Fig. 1(a).

$$EV(G) := \{(e, v) : e \in E(G), v \in V(e)\}.$$

Then we may write the line, subdivision, and middle graphs of G above as follows:

$L(G) = (E(G), EE(G))$,
$S(G) = (V(G) \cup E(G), EV(G))$,
$M(G) = (V(G) \cup E(G), EE(G) \cup EV(G))$.

Let $\mathcal{T}(G)$ denote the set of spanning trees of a graph G and $t(G) = |\mathcal{T}(G)|$. Hence $t(G)$ denotes the number of spanning trees of G. A natural question is to consider the relation between the numbers of spanning trees of G and $L(G)$, $S(G)$, or $M(G)$. For the subdivision graph $S(G)$, it is well known that $t(S(G)) = 2^{m-n+1}t(G)$ [2]. The enumerative problem of spanning trees of the line graph has extensively been studied. We list some related results as follows.

Theorem 1 ([1,3,7,10]). *Suppose G is a Δ-regular graph with n vertices and m edges. Then*

$$t(L(G)) = 2^{m-n+1}\Delta^{m-n-1}t(G). \tag{1}$$

The author of the current paper generalized the result above as follows.

Theorem 2 ([11]). *Suppose G is a connected graph with n vertices and m edges and G' the revision of G. Then*

$$t(L(G')) = 2^{m-n+1}\Delta^{m+s-n-1}t(G), \tag{2}$$

where Δ is the maximum degree of G and s is the number of vertices of degree one in G'.

Dong and Yan [4] used a combinatorial method to obtain a more general result than that in Theorems 1 and 2 as follows.

Theorem 3 ([4]). *Suppose that G is a connected graph with vertex set $V(G)$ and edge set $E(G)$. Then*

$$t(L(G)) = \prod_{v \in V(G)} d(v)^{d(v)-2} \sum_{T \in \mathcal{T}(G)} \prod_{e \in E(G)-E(T)} \left(d(u_e)^{-1} + d(v_e)^{-1}\right), \tag{3}$$

where $d(v) = d_G(v)$, $e = (u_e, v_e)$, and $\mathcal{T}(G)$ denotes the set of spanning trees of G.

The theorem above results in the following corollary.

Corollary 1 ([4]). *Suppose $G = (U, V; E)$ is a connected bipartite graph with n vertices and m edges such that $d(x) \in \{1, a\}$ for all $x \in U$ and $d(y) \in \{1, b\}$ for all $y \in V$, where $a \geq 2$ and $b \geq 2$. Then*

$$t(L(G)) = a^{(a-2)n_1}b^{(b-2)n_2}(a^{-1} + b^{-1})^{m-n+1}t(G), \tag{4}$$

where n_1 is the number of vertices x in U with $d(x) = a$ and n_2 is the number of vertices y in V with $d(y) = b$.

Recently, Gong and Jin [5] used the generalized Wye-Delta transform to obtain a result equivalent to Theorem 3.

For the enumerative problem of spanning trees of the middle graph, by a method based on graph spectra, Huang and Li [6] obtained the following result.