Applied Mathematics and Computation 307 (2017) 329–341

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An efficient parareal algorithm for a class of time-dependent problems with fractional Laplacian

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ARTICLE INFO

Keywords:
Parareal algorithm
Fractional Laplacian
Implicit-explicit Euler method
3rd-order SDIRK method
Convergence analysis

ABSTRACT

Time-dependent diffusion equations with fractional Laplacian have received considerable attention in recent years, for which numerical methods play an important role because a simple and analytic solution is often unavailable. We analyze in this paper a *parareal* algorithm for this kind of problem, which realizes parallel-in-time computation. The algorithm is iterative and uses the 3rd-order SDIRK (singly diagonally implicit Runge-Kutta) method with a small step-size Δt as the $\mathcal{F}$-propagator and the implicit-explicit Euler method with a large step-size ΔT as the $\mathcal{G}$-propagator. The two step-sizes satisfy $\Delta T/\Delta t = J$ with $J \geq 2$ being an integer. Using the implicit-explicit Euler method as the $\mathcal{G}$-propagator potentially improves the parallel efficiency, but complicates the convergence analysis. By employing some technical analysis, we provide a sharp estimate of the convergence rate, which is independent of the mesh ratio J and the distribution of the eigenvalues of the coefficient matrix. An extension of the results to problems with time-periodic conditions is also given. Several numerical experiments are carried out to verify the theoretical results.

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1. Introduction

In this section, we introduce the time-dependent diffusion problems with fractional Laplacian studied in this paper and the parareal algorithm.

1.1. The diffusion problem with fractional Laplacian

In recent years, there is a growing interest in the study of partial differential equations (PDEs) with fractional Laplacian. These equations have a wide application in many fields, such as financial mathematics [6], image processing [14,16], turbulence [2] and many others. We consider in this paper a class of representative PDEs in this field:

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - d(-\Delta)^{\alpha} u(x,t) + f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

where $d > 0$ and $\alpha \in (0, 1)$. As adopted in [1,3–5,19,20,26,29,30,37], here the symmetric spatial fractional Laplacian operator $(-\Delta)^{\alpha}$ is defined through the eigenfunction expansion on a finite-size spatial domain (see Definition 1 in [20]). We remark that it is entirely possible to define the fractional Laplacian operator through the Fourier transform on the whole

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http://dx.doi.org/10.1016/j.amc.2017.02.012
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spatial domain [18,31]. For both definitions, the idea of *matrix transform* proposed in [19] provides efficient spatial discretization [24,36,37]. The spatial discretization based on the matrix transform idea of the fractional Laplacian $(-\Delta)^{\alpha}$ is obtained by first finding a matrix representation $\mathbf{A}$ of the regular (negative) Laplacian operator $-\Delta$ (whether it is through finite difference, finite element, or finite volume) and then raising it to the same fractional power $\mathbf{A}^{\alpha}$[1]. Precisely, for (1.1), by introducing a mesh with m nodes and denoting the value of $u(x, t)$ at the i-th node x_i by $\mathbf{u}_i(t)$, this treatment yields approximation $\mathbf{u}(t) \approx (\mathbf{u}(\mathbf{x_1}, t), \mathbf{u}(\mathbf{x_2}, t), \ldots, \mathbf{u}(\mathbf{x_m}, t))^{\top}$, where $\mathbf{u}(t)$ is the solution of the following ODE system

$$\mathbf{u}'(t) + (\mathbf{A} + d\mathbf{A}^{\alpha})\mathbf{u}(t) = F(t), \quad t \in (0, T), \tag{1.2}$$

where $F(t)$ contains the information of the source term f and the boundary values.

The coefficient matrix $\mathbf{A}$ in (1.2) can be derived, for example, through the centered finite difference scheme subjected with Neumann boundary condition ($\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0$ on $\partial\Omega$), as

$$\begin{cases} \mathbf{A} = \Lambda_{\Delta x} := \frac{1}{\Delta x^2} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{m\times m}, & \text{1-D case,} \\ \mathbf{A} = I \otimes \Lambda_{\Delta x} + \Lambda_{\Delta x} \otimes I, & \text{2-D case,} \\ \mathbf{A} = I \otimes I \otimes \Lambda_{\Delta x} + I \otimes \Lambda_{\Delta x} \otimes I + \Lambda_{\Delta x} \otimes I \otimes I, & \text{3-D case,} \end{cases} \tag{1.3}$$

where Δx denotes the step-size and m denotes the number of nodes in the spatial grid. We can also consider other boundary conditions and other spatial discretizations, such as finite element and finite volume, etc. For the chosen spatial discretization, we emphasize the following two points:

- the matrix $\mathbf{A}$ is sparse and SPD (symmetric positive definite);
- $\lambda \in \sigma(\mathbf{A}) \Leftrightarrow \lambda^{\alpha} \in \sigma(\mathbf{A}^{\alpha})$, where $\sigma(\mathbf{A})$ denotes the spectrum of $\mathbf{A}$.

Note that, in spite of the sparsity of $\mathbf{A}$, the matrix $\mathbf{A}^{\alpha}$ is dense. This implies that it would be very time-consuming to solve (1.2), if the number of spatial grids is large (i.e., the size of the matrix $\mathbf{A}$ is huge). Therefore, the ODE system (1.2) presents the fundamental challenges for numerical investigations of the time-dependent diffusion problems with fractional Laplacian.

1.2. The parareal algorithm

To accelerate the numerical computation of (1.2), we can consider the so-called *parareal* algorithm proposed by Lions, Maday and Turinici [21]. The algorithm starts from splitting the whole time-interval $[0, T]$ into N large subintervals $[T_n, T_{n+1}]$ as $[0, T] = \cup_{n=0}^{N-1}[T_n, T_{n+1}]$ with large step-size ΔT and further dividing each large subinterval $[T_n, T_{n+1}]$ into J small intervals as $[T_n, T_{n+1}] = \cup_{j=0}^{J-1}[T_{n+j/J}, T_{n+(j+1)/J}]$ with small step-size Δt. The two step-sizes satisfy $\Delta T/\Delta t = J$ with $J \geq 2$ being an integer. Then, we choose two numerical methods, denoted by $\mathcal{G}$ and $\mathcal{F}$, to propagate the initial solution $\mathbf{u}_n^k$ on each large subinterval $[T_n, T_{n+1}]$ with step-sizes ΔT and Δt, respectively. The last step is to use a novel correction scheme to improve the approximate solutions on the coarse time-grids $\{T_n\}_{n=1}^{N}$, as

$$\mathbf{u}_{n+1}^{k+1} = \mathcal{G}\left(T_n, \mathbf{u}_n^{k+1}, \Delta T\right) + \mathcal{F}^{J}\left(T_n, \mathbf{u}_n^{k}, \Delta t\right) - \mathcal{G}\left(T_n, \mathbf{u}_n^{k}, \Delta T\right), \tag{1.4}$$

where $n = 0, 1, \ldots, N-1$, $k \geq 0$ denotes the iteration index and for $k = 0$ the initial guess $\{\mathbf{u}_n^0\}_{n=1}^{N}$ are chosen randomly (or generated by a cheap numerical method). Here, $\mathcal{G}\left(T_n, \mathbf{u}_n^k, \Delta T\right)$ denotes the numerical solution of (1.2) at $t = T_{n+1}$ produced by using the numerical method $\mathcal{G}$ with step-size ΔT and initial value $\mathbf{u}_n^k$ at $t = T_n$. For example, if we use the Backward-Euler method as the $\mathcal{G}$-propagator, $\mathcal{G}\left(T_n, \mathbf{u}_n^k, \Delta T\right)$ denotes the solution $\widetilde{\mathbf{u}}$ of the following linear equations

$$\widetilde{\mathbf{u}} + \Delta T[(\mathbf{A} + d\mathbf{A}^{\alpha})\widetilde{\mathbf{u}}] = \mathbf{u}_n^k + \Delta T F(T_n + \Delta T) \left(\Leftrightarrow [I + \Delta T(\mathbf{A} + d\mathbf{A}^{\alpha})]\widetilde{\mathbf{u}} = \mathbf{u}_n^k + \Delta T F(T_{n+1})\right). \tag{1.5}$$

In (1.4), $\mathcal{F}^{J}\left(T_n, \mathbf{u}_n^k, \Delta t\right)$ denotes the numerical solution of (1.2) at $t = T_{n+1}$ by running J steps of the fine propagator $\mathcal{F}$ with small step-size Δt and initial value $\mathbf{u}_n^k$ at $t = T_n$.

It is clear that, upon convergence we have $\mathbf{u}_{n+1}^{\infty} = \mathcal{F}^{J}(T_n, \mathbf{u}_n^{\infty}, \Delta t)$, i.e., the approximate solution of (1.2) at the coarse grids $\{T_n\}_{n=1}^{N}$ will have achieved the accuracy of the fine propagator $\mathcal{F}$ with small step-size Δt. In the past few years, the parareal algorithm and its variants have been used in many fields, such as structural (fluid)-dynamic simulations [7,11], optimal control [8,23,25], Hamiltonian computation [9,10,15], turbulent plasma simulations [27,28], singularly perturbed problems [22] and time-periodic problems [13,32,33], etc. In spite of numerous successful applications as mentioned, the results about the convergence properties of this algorithm are less known in the literature. For the scalar linear model problem $\mathbf{u}'(t) = \lambda \mathbf{u}(t) + f$ with $\lambda \in \mathbb{C}^{-}$, Gander and Vandewalle [12] systematically analyzed the convergence rate of this algorithm.

[1] The advantage of this idea lies in that the matrix-vector product $\mathbf{A}^{\alpha}\mathbf{b}$ can be approximated by many existing and fruitful numerical methods, such as the contour integral method [3], the Lanczos method [30,37] and the preassigned poles and interpolation nodes method [3,30].

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