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Unconditional superconvergence analysis for nonlinear hyperbolic equation with nonconforming finite element

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ABSTRACT

Nonlinear hyperbolic equation is studied by developing a linearized Galerkin finite element method (FEM) with nonconforming EQ_1^{rot} element. A time-discrete system is established to split the error into two parts which are called the temporal error and the spatial error, respectively. The temporal error is proved skillfully which leads to the analysis for the regularity of the time-discrete system. The spatial error is derived τ-independently with order $O(h^2 + h\tau)$ in broken H^1-norm. The final unconditional superclose result of u with order $O(h^2 + \tau^2)$ is deduced based on the above achievements. The two typical characters of this nonconforming EQ_1^{rot} element (see Lemma 1 below) play an important role in the procedure of proof. At last, a numerical example is provided to support the theoretical analysis. Here, h is the subdivision parameter, and τ, the time step.

1. Introduction

Consider the following nonlinear hyperbolic equation:

$$\begin{cases} u_{tt} - \nabla\cdot(a(u)\nabla u) = f(u), & (X,t) \in \Omega\times(0,T], \\ u = 0, & (X,t) \in \partial\Omega\times(0,T], \\ u(X,0) = u_0(X), \quad u_t(X,0) = u_1(X), & X\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a rectangle with the boundary $\partial\Omega$, $0 < T < \infty$ and $X = (x, y)$, $a(u)$, $f(u)$, $u_0(X)$ are known smooth functions. Assume that $a(u)$ is twicely continuously differentiable with respective to u, $0 < a_0 \le a(u) \le a_1$ for certain positive constants a_0, a_1. In addition, both $a_{uu}(u)$ and $f(u)$ are globally Lipschitz continuous in u.

In physics, the hyperbolic equations are important partial differential equations in describing the propagation of sound and electromagnetic wave and so on. Various numerical methods have been investigated on such problems (refer to [1–10] for linear cases). Indeed, a lot of studies on the nonlinear hyperbolic equation, from both theoretical and practical point of view, are useful in solving the problems of nonlinear vibration and permeation fluid mechanics. For instance, Zhou et al. [11] and Chen [12] discussed the H^1-Galerkin expanded mixed FEM and usual mixed FEM respectively, and both of them arrived at optimal error estimates. Shi and Li [13] obtained the superclose estimate for the nonlinear hyperbolic equations with nonlinear boundary condition through interpolation, and the global superconvergence result was deduced based on the interpolated postprocessing technique. The Galerkin alternating-direction procedure for a kind of three-dimensional nonlinear hyperbolic equation was considered in [14] and the error estimates in H^1 norm and L^2

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norm were demonstrated by using a priori estimate. However, to study the time-dependent optimal error estimates for a nonlinear physical system, the boundedness of numerical solution U_h^n in L^∞-norm or a stronger norm is often required, and the inverse inequality is usually employed to deal with such issue, which will result in some time-step restrictions, such as $\tau = O(h), h^r = O(\tau)(1 \le r \le k+1, k \ge 0)$ and $\tau = O(h^2)$ in [12] and [14], respectively. In fact, in the researches of other nonlinear evolution equations also need some restrictions of τ, such as nonlinear parabolic equation [15,16], nonlinear Sobolev problems [17,18], nonlinear Schrödinger equations [19,20] and Navier–Stokes equations [21,22], and so on.

To overcome such deficiency, [23] constructed a corresponding time-discrete system to split the error into two parts, the temporal error and the spatial error. Then the spatial error reduces to the unconditional boundedness of numerical solution in L^∞-norm. Subsequently, this so-called splitting technique was also applied to other equations [24–29]. In the above studies they only arrived at optimal estimates. Recently, Shi [30] made use of the nature of the equation to get the unconditional superclose for Sobolev equation with conforming mixed FEM. However, as far as we know, there is no consideration about the nonlinear hyperbolic equation.

The main aim of the present work is to discuss the unconditional superconvergence estimate for (1.1) with nonconforming EQ_1^{rot} element [31,32]. Firstly, we develop a linearized FE scheme with second order, which is different from the traditional Crank–Nicolson scheme, and then motivated by the idea of splitting technique in [23–29], a time-discrete system with solution U^n is introduced to split the error $u^n - U_h^n$ into the temporal error $u^n - U^n$ and the spatial error $U^n - U_h^n$. Secondly, we obtain the temporal error which reduces to the regularity of U^n and derive the unconditional superclose result of u in broken H^1-norm with order $O(h^2 + \tau^2)$ by arriving at the spatial error with order $O(h^2 + h\tau)$ directly. At last, some numerical results show the validity of the theoretical analysis. We also point out that the analysis presented herein is also valid to some other nonconforming FEs possessing the two properties in Lemma 1 and the assumption of the condition about $\partial\Omega$ is less stringent here compared with that in [24–29] where it is smooth enough.

Throughout this paper, we denote the natural inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$ and the norm by $\|\cdot\|_0$, and let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Further, we use the classical Sobolev spaces $W^{m,p}(\Omega)$, $1 \le p \le \infty$, denoted by $W^{m,p}$, with norm $\|\cdot\|_{m,p}$. When $p = 2$, we simply write $\|\cdot\|_{m,p}$ as $\|\cdot\|_m$. Besides, we define the space $L^p(a, b; Y)$ with the norm $\|f\|_{L^p(a,b;Y)} = (\int_a^b \|f(\cdot,t)\|_Y^p dt)^{\frac{1}{p}}$, and if $p = \infty$, the integral is replaced by the essential supremum.

2. Nonconforming FE approximation scheme

Let Ω be a rectangle in (x, y) plane with edges parallel to the coordinate axes, Γ_h be a regular rectangular subdivision. Given $K \in \Gamma_h$, set the four vertices and edges be $a_i, i = 1 \sim 4$ and $l_i = \overline{a_i a_{i+1}}, i = 1 \sim 4 \pmod 4$, respectively. Then the EQ_1^{rot} finite element space V_h is defined as [32]:

$$V_h = \{v_h; v_h|_K \in \text{span}\{1, x, y, x^2, y^2\}, \int_F [v_h]ds = 0, F \subset \partial K, \forall K \in \Gamma_h\},$$

where $[v_h]$ stands for the jump of v_h across the edge F if F is an internal edge, and v_h itself if F is a boundary edge. Let I_h: $H^1(\Omega) \to V_h$ be the associated interpolation operator on $V_h, I_K = I_h|_K$ satisfying

$$\int_{l_i} (I_K u - u)ds = 0, i = 1 \sim 4, \int_K (I_K u - u)dX = 0. \tag{2.1}$$

The following lemma plays an important role later and it can be found in [32].

Lemma 1. *If $\psi \in H^1(\Omega)$ and $\varphi \in (H^2(\Omega))^2$, then for all $v_h \in V_h$, there hold*

$$(\nabla_h(\psi - I_h\psi), \nabla_h v_h) = 0, \tag{2.2}$$

and

$$\left| \sum_K \int_{\partial K} \varphi \cdot \vec{n} v_h ds \right| = O(h^2)|\varphi|_2 \|v_h\|_h. \tag{2.3}$$

Here and later, ∇_h denotes the gradient operator defined piecewisely, $(\star, \star)_h = \sum_K \int_K \star \cdot \star dX$ and $\|\cdot\|_h = (\sum_K |\cdot|_{1,K}^2)^{\frac{1}{2}}$ is a norm on V_h.

Let $\{t_n : t_n = n\tau; 0 \le n \le N\}$ be a uniform partition of $[0, T]$ with the time step $\tau = T/N$. We denote $\sigma^n = \sigma(X, t_n)$. For a sequence of functions $\{\sigma^n\}_{n=0}^N$, we describe some of the notations which will be frequently used in this paper.

$$\bar{\partial}_{tt}\sigma^n = \frac{1}{\tau^2}(\sigma^{n+1} - 2\sigma^n + \sigma^{n-1}), \bar{\partial}_t\sigma^n = \frac{1}{2\tau}(\sigma^{n+1} - \sigma^{n-1}), \tilde{\partial}_t\sigma^n = \frac{1}{\tau}(\sigma^n - \sigma^{n-1}),$$
$$\sigma^{n,\frac{1}{4}} = \frac{1}{4}(\sigma^{n+1} + 2\sigma^n + \sigma^{n-1}), \hat{\sigma}^n = \frac{1}{4}(3\sigma^n + 2\sigma^{n-1} - \sigma^{n-2}), \bar{\sigma}^n = \frac{1}{2}(\sigma^n + \sigma^{n-1}),$$

which implies

$$\bar{\partial}_t\sigma^n = \tilde{\partial}_t\bar{\sigma}^{n+1}, \sigma^{n,\frac{1}{4}} = \frac{1}{2}(\bar{\sigma}^{n+1} + \bar{\sigma}^n), \hat{\sigma}^n = \frac{1}{2}(3\bar{\sigma}^n - \bar{\sigma}^{n-1}),$$