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On linear differential equations and systems with reflection☆

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ABSTRACT

In this paper, we develop a theory of linear differential systems analogous to the classical one for ODEs, including the obtaining of fundamental matrices, the development of a variation of parameters formula and the expression of the Green's functions. We also derive interesting results in the case of differential equations with reflection and generalize the Hyperbolic Phasor Addition Formula to the case of matrices.

1. Introduction

In recent years, there have been a number of works concerning the study of differential problems with involutions. In the particular case of the reflection, starting with [5], the computation of Green's functions for differential equations with reflection expanded [6–10,17,18]. This has motivated several applications concerning almost periodic solutions [15,16], the obtaining of eigenvalues and explicit solutions of different problems [13,14] or their qualitative properties [2,4].

On the other hand, what happens in the case of linear differential systems with reflection and constant coefficients has drawn far less attention [1]. The authors intend, in this article, to provide some insight on this question. To do so, in Section 2 we retake the original problem (for differential equations with reflection), providing two interesting results. First, we give an improvement on the general Reduction Theorem –see for instance [9, Theorem 5.1.1], here Theorem 2.1, that reduces the order of the resulting ODE –see Theorem 2.3. Second, we provide an explicit basis of the space of solutions of linear differential equations with reflection and constant coefficients –see Theorem 2.5. This knowledge is fundamental if our intention is to construct a fundamental matrix of the associated homogeneous problem.

This first part of the paper suggests that a similar attempt should be done in the case of systems of differential equations with reflections. Our approach will run in parallel to the classical theory of linear ODEs: construction of a fundamental matrix, description of the method of variation of parameters and, finally, obtaining of the associated Green's function. Unfortunately, this process will not be devoid of difficulties. That is why in Section 3, we will summarize some results concerning matrix functions which will be useful latter on.

We start the study of systems of linear equations with reflection in Section 4. The first thing we do is to *define* what a fundamental matrix is going to be in this setting. This is not obvious, for there are some properties that are satisfied in the case of systems of ODEs which will not apply here. For instance, contrary to our experience, a fundamental matrix *may* be singular at some point of the real line. Once this definition is properly established, it is time to derive the most basic results of the theory: those concerning existence and uniqueness of solution. Existence is derived from the analogous

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result for systems of ODEs –Lemma 4.2, while existence is obtained through the direct construction of a fundamental matrix. This result –Theorem 4.5, arguably one of the main results of the paper, expresses this fundamental matrix as a series of functional matrices. It is only under some extra assumptions that a simpler expression involving hyperbolic trigonometric functions may be found. The rest of the Section consists of rewriting this fundamental matrix in other ways. In order to achieve this, we have to generalize the Hyperbolic Phasor Addition Formula [20, Lemma 1] to the algebra of matrices.

Section 5 concerns the method of variation of parameters. Again, the method differs from the one we have in the case of ODEs. First we show that the classical approach does not work in this setting and then, studying a complementary problem, we arrive to a general method –Theorem 5.6.

Finally, in Section 6 we use the method of variation of parameters to obtain the Green's function both in the initial condition and the two point boundary condition cases. This is a natural generalization of the previous settings when concerning differential equations with reflections. To illustrate this point we recover, as shown in Example 6.7, the known expression of the Green's function for a first order periodic equation with reflection.

2. Differential equations with reflection

Let us introduce some definitions and notations. To start with, consider the differential operator D, the pullback operator of the reflection $\varphi(t)=-t$, denoted by $\varphi^*(u)(t):=u(-t)$, and the identity operator, Id (we will also denote by Id the identity matrix).

Let $T\in(0,+\infty)$ and $I:=[-T,T]$. We now consider the ring $\mathbb{R}[D]$ of polynomials with real coefficients on the variable D and the algebra $\mathbb{R}[D,\varphi^*]$ consisting of the operators of the form

$$L:=\varphi^*P(D)+Q(D), \tag{2.1}$$

where $P(D)=\sum_{k=0}^n b_kD^k,\ Q(D)=\sum_{k=0}^n a_kD^k\in\mathbb{R}[D]$ $(D^0=\mathrm{Id})$, $n\in\mathbb{N},\ a_k,b_k\in\mathbb{R},\ \ k=1,\dots,n$ which act as follows:

$$Lu(t)=\sum_{k=0}^n a_ku^{(k)}(t)+\sum_{k=0}^n b_ku^{(k)}(-t),\ \ t\in I,$$

on any function $u\in W^{n,1}(I)$. The operation in the algebra $\mathbb{R}[D,\varphi^*]$ is the usual composition of operators (most of the time we will omit the composition sign). We observe that $D^k\varphi^*=(-1)^k\varphi^*D^k$ for $k=0,1,\dots,$ which makes it a *noncommutative algebra*. Actually, we have that $P(D)\varphi^*=\varphi^*P(-D)$ for any $P\in\mathbb{R}[D]$.

The following property is crucial for the obtaining of a Green's function.

Theorem 2.1 ([10, Theorem 2.1]). *Take L as defined in (2.1) and define*

$$L_1:=\varphi^*P(D)-Q(-D)\in\mathbb{R}[D,\varphi^*]. \tag{2.2}$$

Then $L_1L=LL_1\in\mathbb{R}[D]$.

Remark 2.2. As it is pointed out in [10], if $L_1L=\sum_{k=0}^{2n}c_kD^k$, then

$$c_k=\begin{cases}0, & k\text{ odd},\\ 2\displaystyle\sum_{l=0}^{\frac{k}{2}-1}(-1)^l(b_lb_{k-l}-a_la_{k-l})+(-1)^{\frac{k}{2}}\left(b_{\frac{k}{2}}^2-a_{\frac{k}{2}}^2\right), & k\text{ even}.\end{cases}$$

If $L=\sum_{i=0}^n(b_i\varphi^*+a_i)D^i$ with $a_n\neq0$ or $b_n\neq0$, we have that $c_{2n}=(-1)^n(b_n^2-a_n^2)$. Hence, if $a_n=\pm b_n$, then $c_{2n}=0$. This shows that composing two elements of $\mathbb{R}[D,\varphi^*]$ we can get another element with derivatives of less order.

We can improve Theorem 2.1 in the following way. Let

$$R(D):=\mathrm{mcd}(P(D),Q(D),P(-D),Q(-D)),\ \ \tilde P=P/R\text{ and }\tilde Q=Q/R.$$

Observe that R is the polynomial constructed from the common roots, according to multiplicity, of $P(D)$, $Q(D)$, $P(-D)$ and $Q(-D)$. $R(D)=R(-D)$, for if λ is an root of $P(D)$ so has to be of $P(-D)$, and so $-\lambda$ has to be a root of $P(D)$. An important consequence of this is that R commutes with φ^*. We now have all it is needed for an improved version of Theorem 2.1.

Theorem 2.3. *Take* L, R, $\tilde P$ *and* $\tilde Q$ *as previously defined and define*

$$\widehat{L}:=\varphi^*\tilde P(D)-\tilde Q(-D)\in\mathbb{R}[D,\varphi^*]. \tag{2.3}$$

Then $\widehat{L}L=L\widehat{L}\in\mathbb{R}[D]$.

Proof.

$$\begin{aligned}\widehat{L}L&=[\varphi^*\tilde P(D)-\tilde Q(-D)][\varphi^*P(D)+Q(D)]=[\varphi^*\tilde P(D)-\tilde Q(-D)][\varphi^*\tilde P(D)+\tilde Q(D)]R(D)\\&=[\tilde P(-D)\tilde P(D)+\varphi^*\tilde P(D)\tilde Q(D)-\varphi^*\tilde Q(D)\tilde P(D)-\tilde Q(-D)\tilde Q(D)]R(D)\\&=[\tilde P(-D)\tilde P(D)-\tilde Q(-D)\tilde Q(D)]R(D).\end{aligned}$$