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Numerical analysis and simulation of a bio-thermal model for the human foot

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ABSTRACT

This paper analyses the numerical convergence and the approximation of the finite element method applied to a biothermal nonlinear model for the bare foot. As far as we know, there is not a previous finite element analysis of this well-known bioheat equation. Thus, this work can be seen as a first step to study the coupling with energy and mass transfer models (water, vapor and gas) at the textiles surrounding the foot. The model is posed as a steady partial differential equation for the temperature field, and a non-linear boundary condition on the external boundary, where heat losses due to convection, radiation and evaporation are considered. The existence and the uniqueness of the solution is proved for the weak formulation and also for its finite element approximation by using arguments of monotone operators. Then, numerical convergence and an a priori error estimates result are obtained. Some numerical simulations are presented to show the accuracy of the numerical method and the behavior of the solution, being qualitatively acceptable and, in some cases, validated against experimentation, being quantitatively correct too. Interesting conclusions are followed from the analysis of the model parameters as well as from the comparison of 2D and 3D solutions.

1. Introduction

It is well known that the global human thermoregulatory system plays an important role in clinical diagnosis, medical therapies or physiological processes involved in many human body disorders (see, for instance, [8–10,21] and references therein). From non serious problems, as thermal comfort [20], to hyperthermia therapy used in critical cancer diseases [14], human thermoregulation research has been an important research topic during the last fifty years (see, for instance, [1,13,15] and references therein). In fact, the mathematical modeling of this thermal problem has a long history of almost three centuries [21], but it is after the pioneering work of Pennes's [18] that most of the details of the models related to the bioheat equation, known as Pennes equation, are provided, which will also be the starting point here. In addition, there are many papers applying numerical methods for solving the Pennes bioheat transfer equation such as finite differences, finite elements, finite volumes, boundary elements, spectral methods and so on (see the very complete numerical survey given in

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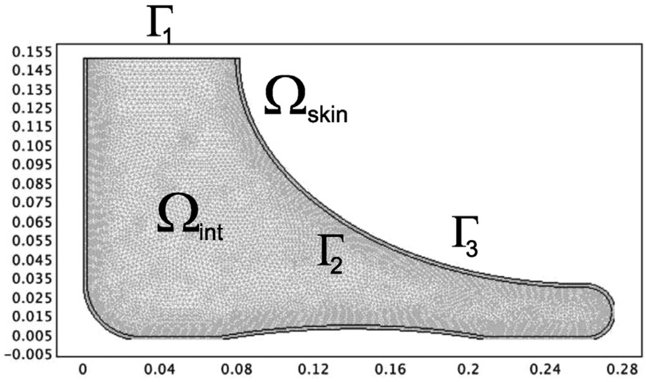

Fig. 1. Physical setting including internal tissue and skin (in meters) and mesh (12,920 triangular finite elements of 1 mm of characteristic size, approximately).

[7] and references therein). However, only very few works deal exclusively with the thermal modeling and the numerical simulation of the bare human foot. For instance, an analytical mathematical solution for a simplified one-dimensional case is given in the recent work of Ma et al. [16], and finite element approximations can be seen in the papers by Covill et al. [5,6]. Anyway, literature sources do not show such a complete numerical analysis of this type of approximations, providing the existence and uniqueness of the finite elements solution, convergence and error estimates on a more mathematically rigorous basis.

Thus, in this work we consider a model for the bare foot based on the metabolic bioheat equation, incorporating the blood flow as a heat source and the effects of sweating using semi empirical models. Since this equation is linear, the main difficulty of the analysis concerns to the boundary condition related to the boundary skin-environment, because the radiation term is assumed to be nonlinear. The aim consists of fitting the numerical approximation of those thermal equations and the parameters involved in the bare foot model as a previous study, before its coupling to a mass and energy transfer models at a textile that surrounds the foot is developed. The future numerical solution of this more complex global problem may be a very interesting tool for virtual designs for the sport and health footwear industries.

This paper is then outlined in the following form. In Section 2 we describe the mathematical model, we derive its variational formulation and we prove the existence and uniqueness of the solution by using monotone operators. Then, in Section 3, we introduce numerical approximations applying the finite element method. Moreover, some a priori error estimates are obtained, from which the convergence of the algorithm is deduced under adequate additional regularity conditions. Finally, Section 4 briefly describes the implemented numerical algorithm and some numerical results which show the accuracy of the finite element approximation and the behavior of the solution. In addition, modeling and numerical simulations are validated through experimental data.

2. Thermal model and weak formulation

Usually, a large simplification of the parts of the foot (bones, muscles, tendons, etc.) is made in order to reduce the complex internal composition to a single tissue, where the thermal and biological properties are averaged. However, the consideration of the foot skin as another tissue can be very important because its relevance on the environmental boundary conditions. Therefore, let denote by $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the domain occupied by the bare human foot, assumed to be divided into two different disjoint parts, Ω_{int} and Ω_{skin}, defining the internal tissue and the skin, respectively. Moreover, let $\boldsymbol{\nu}$ be the outward unit vector normal to the boundary $\Gamma = \partial\Omega$. This boundary Γ is assumed to be splitted into three disjoint parts Γ_1, Γ_2 and Γ_3. Boundary Γ_1 denotes the part of the foot joined to the rest of the leg, boundary Γ_2 represents the common boundary between Ω_{int} and Ω_{skin}, and boundary Γ_3 is the part of the skin in contact with the environment, where the heat losses are produced (see Fig. 1).

Let us denote by T the unknown temperature of the foot. Therefore, according to Wu et al. [22] and many other references related to Pennes equation, the bioheat transfer which models the bare foot is written as follows:

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \rho_b c_b \omega_b (T_b - T) + Q_m \quad \text{in} \quad \Omega, \tag{1}$$

where ρ is the density of the tissue, c is the specific heat of the tissue, k is the thermal conductivity which has different values in the internal tissue and the skin, ρ_b is the blood density, c_b is the blood specific heat, ω_b is the blood perfusion volumetric rate, T_b is the blood temperature (assumed to be constant for simplicity) and Q_m is the metabolic heat of the foot tissues.

Transient Pennes bioheat Eq. (1) plays an important role in hyperthermia cancer therapy (see, for instance, [14]) because the living tissues are subjected to sudden changes of the heat sources. However, in this study the foot will be only affected