Applied Mathematics and Computation 305 (2017) 117–135

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A porous thermoelastic problem: An a priori error analysis and computational experiments

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ARTICLE INFO

Keywords:
Thermoelasticity
Porosity
Finite element approximations
Error estimates
Numerical simulations

ABSTRACT

In this paper, a porous thermoelastic problem is numerically considered. The variational formulation is written as a coupled system of two hyperbolic equations for the displacement and the porosity fields and a parabolic equation for the temperature field. An existence and uniqueness result as well as an energy decay property are recalled. Then, fully discrete approximations are introduced by using the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the first-order time derivatives. A priori error estimates are proved, from which the linear convergence is deduced under some additional regularity conditions. Finally, some one- and two-dimensional numerical simulations are presented to show the accuracy of the approximation and the behavior of the solution.

1. Introduction

The mechanics of porous materials is of great interest to the field of solids mechanics, foundation engineering as well as powder technology. Since the first theory of granular materials formulated in [1] (see also [2]), many authors considered nonlinear and linear theories of elastic materials with voids. Their main idea was to introduce a new kinematic variable which represented the volume fraction field. In particular, Cowin and Nunziato established the linear theory in [3], which was extended to include the thermal effects by Ieşan in [4]. The literature dealing with thermoelastic materials with voids is rather extensive (see, for instance, [5–8], the monograph [9] and the references cited therein).

Recently, Casas and Quintanilla introduced in [10] a theory for the behavior of porous solids such that the matrix material was elastic and the interstices were void of material, which was a generalization of the classical theory of elasticity. Moreover, two kinds of dissipation process were considered, involving the viscosity type in the porous structure and the thermal dissipation, because when only thermal damping or only porous damping are assumed the solutions decay slowly (see [11]). Therefore, in this paper we consider this problem from the numerical point of view, proving some a priori error estimates, from which the linear convergence is deduced under suitable additional regularity conditions, and performing some numerical simulations.

The outline of this paper is as follows. In the first section, the mathematical model and its variational formulation are presented. An existence and uniqueness result and an energy decay property, obtained proceeding as in [10], are recalled. Then, fully discrete approximations are introduced in the second section by using the finite element method for the spatial

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http://dx.doi.org/10.1016/j.amc.2017.01.070
0096-3003/

approximation and the backward Euler scheme for the discretization of the time derivatives. A priori error estimates are obtained from which the linear convergence is deduced under suitable additional regularity conditions. Finally, in the last section some one- and two-dimensional numerical examples are shown to demonstrate the accuracy of the algorithm and the behavior of the solution.

2. The model and its variational formulation

In this section, we present briefly the model, the required assumptions and the variational formulation of the mechanical problem (details can be found in [9,10]).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the elastic body and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body $\Gamma = \partial\Omega$ is assumed to be Lipschitz, with outward unit normal vector $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$, and decomposed into two measurable parts Γ_D and Γ_F such that $\text{meas}(\Gamma_D) > 0$. The body is being acted upon by a volume force with density $\boldsymbol{f}_0$, it is clamped on Γ_D and surface tractions with density $\boldsymbol{f}_F$ act on Γ_F. Moreover, let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t.

According to [10], an elastic body is considered assuming that the matrix material is elastic and that interstices are void of material. This constitutes a generalization of the classical theory of elasticity (see, for instance, [3,6]).

Therefore, denote by $\mathcal{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ and let $\boldsymbol{\sigma} \in \mathbb{S}^d$, $\boldsymbol{u} \in \mathbb{R}^d$ and ϕ, $\theta \in \mathbb{R}$ be the stress field, the displacement field of the solid elastic material, the porosity (or volume fraction) and the difference of the temperature between the actual state and a reference temperature, respectively.

The constitutive law is then written as follows (see, for instance, [12]),

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{B}\theta + \mathcal{D}^*\phi,$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d$ is the well-known linearized strain tensor given by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

$\mathcal{A} = (a_{ijkl})_{i,j,k,l=1}^d$ denotes the fourth-order elasticity tensor, $\mathcal{B} = (b_{ij})_{i,j=1}^d$ is the second-order thermal expansion tensor and $\mathcal{D}^* = (d_{ij}^*)_{i,j=1}^d$ is the second-order porosity tensor.

In order to simplify the writing of the problem and the calculations developed in the next section, we assume that the material is isotropic and so the previous constitutive equation can be written as

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathcal{I} - b\theta\mathcal{I} + d^*\phi\mathcal{I},$$

where tr is the trace operator, $\mathcal{I}$ represents the identity matrix, λ and μ denote the classical Lame's coefficients, b is a thermal expansion coefficient and d^* is a porosity coefficient.

We turn now to describe the boundary conditions.

On the boundary part Γ_D we assume that the body is clamped and thus the displacement field vanishes there (and so $\boldsymbol{u} = 0$ on $\Gamma_D \times (0, T)$). Moreover, since the density of traction forces $\boldsymbol{f}_F$ is applied on the boundary part Γ_F, it follows that $\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_F$ on $\Gamma_F \times (0, T)$. For the sake of simplicity in the presentation of the problem we assume homogenous boundary conditions for the porosity and temperature fields and so, ϕ and θ vanish on the whole boundary Γ, but we point out that other boundary conditions could be used with some minor modifications in the variational and numerical analysis presented below.

Therefore, following [10] the mechanical problem of an isotropic porous thermoelastic body is written as follows.

Problem P. *Find the stresses* $\boldsymbol{\sigma} : \overline{\Omega} \times [0, T] \to \mathbb{S}^d$, *the displacements* $\boldsymbol{u} : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$, *the porosity* $\phi : \overline{\Omega} \times [0, T] \to \mathbb{R}$ *and the temperature* $\theta : \overline{\Omega} \times [0, T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathcal{I} - b\theta\mathcal{I} + d^*\phi\mathcal{I} \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$\rho\boldsymbol{u}_{tt} - \text{Div}\,\boldsymbol{\sigma} = \boldsymbol{f}_0 \quad \text{in} \quad \Omega \times (0, T), \tag{2}$$

$$J\phi_{tt} = \alpha\Delta\phi - d^*\text{div}\,(\boldsymbol{u}) - \xi\phi + m\theta - \tau\phi_t \quad \text{in} \quad \Omega \times (0, T), \tag{3}$$

$$c\theta_t = k^*\Delta\theta - b\text{div}\,(\boldsymbol{u}_t) - m\phi_t \quad \text{in} \quad \Omega \times (0, T), \tag{4}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_D \times (0, T), \tag{5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0, T), \tag{6}$$

$$\phi = 0, \quad \theta = 0 \quad \text{on} \quad \Gamma \times (0, T), \tag{7}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \phi(\boldsymbol{x}, 0) = \phi_0(\boldsymbol{x}), \quad \theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in \Omega, \tag{8}$$