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Constructing a Godunov-type scheme for the model of a general fluid flow in a nozzle with variable cross-section

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ABSTRACT

A Godunov-type scheme for the model of a general fluid flow in a nozzle with variable cross-section is presented. The model has the form of a nonconservative system of balance laws, which poses many challenging questions for study. First, exact Riemann solvers in computing form are described in both subsonic and supersonic regions. Second, the computable exact solutions of local Riemann problem are incorporated into a Godunov-type scheme. Third, the scheme is shown to be well-balanced in the sense that it can capture exactly stationary waves. Finally, numerical tests for data belong to both subsonic and supersonic regions are presented. These tests show that the scheme has a very fine accuracy. Especially, the scheme can give very good approximations to the exact solutions even in the resonant phenomenon where the solution contains three waves of the same speed.

1. Introduction

Nonconservative systems of balance laws have many important applications. In particular, they are used to model multi-phase flows. Nonconservative terms in this kind of systems often causes lots of inconvenience for standard numerical schemes. For example, the errors may become larger as the mesh sizes get smaller, numerical oscillations of the approximate solutions can still remain when the CFL number is decreasing, etc. Therefore, the study of numerical approximations for nonconservative balance laws has attracted many authors for many years.

Our aim in this paper is to construct a Godunov-type numerical scheme for the following nonconservative hyperbolic system: the model of a fluid flow in a nozzle with variable cross-section

$$\begin{aligned} \partial_t(a\rho) + \partial_x(a\rho u) &= 0, \\ \partial_t(a\rho u) + \partial_x(a(\rho u^2 + p)) &= p\partial_x a, \\ \partial_t(a\rho e) + \partial_x(au(\rho e + p)) &= 0, \quad x \in \mathbb{R}, \quad t > 0, \end{aligned} \tag{1.1}$$

where $\rho = \rho(x,t)$, $\varepsilon = \varepsilon(x,t)$, and $p = p(x,t)$ denote the thermodynamical variables: density, internal energy, and the pressure, respectively; $u = u(x,t)$ is the velocity, and $e = e(x,t) = \varepsilon + u^2/2$ is the total energy. The function $a = a(x) > 0, x \in \mathbb{R}$, is the cross-sectional area.

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Basic properties of the system (1.1) was investigated by Thanh [35], where the following trivial equation

$$\partial_t a = 0 \tag{1.2}$$

is supplemented. Observe that solutions of nonconservative systems of balance laws can be understood in the sense of *nonconservative products*, see [14].

In our earlier work [13], a Godunov-type scheme for the *isentropic* model of a fluid flow in a nozzle with variable cross-section was constructed. The development in this work is to build a Godunov-type scheme for the full model (1.1) for a general fluid flow. First, we describe *computing* exact Riemann solutions. Then, the computing solutions of local Riemann problem is incorporated into the Godunov-type scheme. We construct the scheme for data both subsonic and supersonic regions. It is interesting that the scheme is well-balanced, in the sense that it can capture exactly stationary waves. Numerical tests show that the scheme can work and provides us with very fine accuracy for a quite large *CFL* number, says, $CFL = 0.9$. Furthermore, the Godunov-type scheme can resolve the difficulty for certain existing schemes when dealing with the resonant cases: approximate solutions are demonstrated to be convergent to the exact solutions even when the exact solution expands on both subsonic and supersonic region. In particular, in the resonant case where an exact Riemann solution contains three waves of the same zero speed, our scheme can give good approximations to the exact solutions.

There are many works in the literature concerning the theory and the Riemann problem for hyperbolic systems of balance laws in nonconservative form. The reader is referred to [25,28,35] for the Riemann problem for the model of a fluid flow in a nozzle with discontinuous cross-section, to [8,26,27,31] for the Riemann problem for the shallow water equations with discontinuous topography; to [32,34,36] for the Riemann problem for two-phase flow models, and to [17,21,22] for the Riemann problem for other nonconservative models.

Many approaches for numerical approximations of conservation laws with nonconservative source terms have been proposed: well-balanced schemes, path-conservative schemes, ADER schemes, etc., see [3,6,7,11,12,16,19,29,42]. In particular, well-balanced schemes for the model (1.1) and and the shallow water equations with variable topography were constructed in [4,9,18,23,24,27,37,38]. Numerical schemes for multi-phase flow models were presented in [1,5,20,30,39,40]. Godunov-type schemes for hyperbolic systems of balance laws in nonconservative form were studied in [2,9,10,22,27,32,33]. We refer to the books [15,41] for Rieman solvers and numerical schemes for the gas dynamics equations and hyperbolic systems of conservation laws.

This paper is organized as follows. Section 2 provides us with basic concepts and properties of the model (1.1) supplemented with the trivial equation (1.2). Section 3 is devoted to the revisited Riemann problem. A Godunov-type scheme for the model (1.1) will be constructed in Section 4. Numerical tests are given in Section 5. Finally, we draw concluding remarks in Section 6.

2. Preliminaries

2.1. Nonstrict hyperbolicity

To deal with the nonconservativeness of the system (1.1), we supplement it with the trivial equation (1.2). We therefore have the following system of balanced laws:

$$\begin{aligned}
&\partial_t(a\rho) + \partial_x(a\rho u) = 0,\\
&\partial_t(a\rho u) + \partial_x(a(\rho u^2 + p)) = p\partial_x a,\\
&\partial_t(a\rho e) + \partial_x(au(\rho e + p)) = 0,\\
&\partial_t a = 0, \quad x \in \mathbb{R}, \quad t > 0.
\end{aligned} \tag{2.1}$$

For simplicity, we assume that the fluid is polytropic and ideal so that the equation of state is given by

$$p = (\gamma - 1)\rho\varepsilon, \quad \gamma > 1. \tag{2.2}$$

Let us take (p, S) as two independent thermodynamic variables, where S is the entropy. Then, the polytropic ideal gas equation of state can be represented by

$$\rho = \rho(p, S) = \left(\frac{p}{\gamma - 1}\exp\left(\frac{S_* - S}{C_v}\right)\right)^{1/\gamma}, \tag{2.3}$$

where $C_v = R/(\gamma - 1)$ and R is the specific gas constant.

For convenient, we denoted $\kappa(S)$ by

$$\kappa(S) := (\gamma - 1)\exp\left(\frac{S - S_*}{C_v}\right).$$

Then, (2.3) can be written by

$$\rho = \rho(p, S) = p^{\frac{1}{\gamma}}(\kappa(S))^{-\frac{1}{\gamma}}.$$