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On coindices of graphs and their complements

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ABSTRACT

It was recently discovered (Gutman et al., 2015) that the coindex of the first Zagreb index of any graph is equal to the coindex of the complement of this graph. We examine the possibility that an analogous equality holds for other topological indices. It is shown that there are several classes of graphs and classes of topological indices for which such a relation is obeyed.

1. Introduction

In this paper, we are concerned with simple graphs, that is graphs without multiple, weighted, or directed edges, and without self-loops. Let G be such a graph, possessing n vertices and m edges. Its vertex and edge sets are $\mathbf{V}(G)$ and $\mathbf{E}(G)$, respectively. The edge of G, connecting the vertices u and v of the graph G will be denoted by uv.

Let f_v be be some quantity that is uniquely determined by the graph G and its vertex v. Typical such quantities are the degree and eccentricity of the vertex v, or some property of the subgraph $G - v$. In the recent mathematical, network-theoretical, and chemical literature, countless graph invariants have been and are being examined, usually referred to as "*topological indices*" or "*structure descriptors*" [2–4,8,9,16,17]. Many of these are of the form

$$\mathcal{I} = \mathcal{I}(G) = \sum_{uv \in \mathbf{E}(G)} g(f_u, f_v) \tag{1}$$

where $g(x, y)$ is a pertinently chosen function, stipulated to satisfy the condition $g(x, y) = g(y, x)$.

For example, if f_v is the degree of the vertex v and if $g(x, y) = x + y$, then $\mathcal{I}(G)$ is the classical "*first Zagreb index*" $M_1(G)$ [1,7,10–12,14,15]. Note that M_1 is related also to the (general) zero-th Randić index [13]. Thus,

$$M_1(G) = \sum_{uv \in \mathbf{E}(G)} \big[d(u) + d(v)\big].$$

Here and later, $d(v)$ denotes the degree of the vertex v (in the graph G).

In 2008, Došlić [5] put forward the concept of first Zagreb coindex, defining it via

$$\overline{M}_1(G) = \sum_{uv \notin \mathbf{E}(G)} \big[d(u) + d(v)\big].$$

In view of Eq. (1), in analogy to the first Zagreb coindex, one could conceive a general graph coindex, defined as

$$\overline{\mathcal{I}} = \overline{\mathcal{I}}(G) = \sum_{uv \notin \mathbf{E}(G)} g(f_u, f_v). \tag{2}$$

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In what follows, $\overline{G}$ stands for the complement of the graph G, i.e., $\mathbf{V}(\overline{G}) = \mathbf{V}(G)$, and $uv \in \mathbf{E}(\overline{G})$ if and only if $uv \notin \mathbf{E}(G)$. The number of edges of $\overline{G}$ will be denoted by $\overline{m}$. The invariant f, pertaining to the vertex v of the graph $\overline{G}$ will be denoted by $\overline{f}_v$. In particular, $\overline{d}(v)$ is the degree of the vertex v of the graph $\overline{G}$. Recall that $\overline{m} = \binom{n}{2} - m$ and $\overline{d}(v) = n - 1 - d(v)$.

In the study of the first Zagreb coindex, the following remarkable result was discovered [11]. For any (n, m)-graph,

$$\overline{M}_1(G) = 2m(n-1) - M_1(G)$$
$$\overline{M}_1(\overline{G}) = 2m(n-1) - M_1(G)$$

i.e., the equality

$$\overline{M}_1(G) = \overline{M}_1(\overline{G}) \tag{3}$$

is satisfied by any graph G.

Bearing in mind Eqs. (1) and (2), the obvious question is whether there are other topological indices, for which the direct generalization of the relation (3) holds, that is

$$\overline{\mathcal{I}}(G) = \overline{\mathcal{I}}(\overline{G}). \tag{4}$$

In the general case, equality in (4) is violated. Nevertheless, there are several classes of graphs and classes of topological indices for which the relation (4) is obeyed. The aim of the present paper is to establish conditions under which the identity (4) holds.

2. Simple results

A graph is said to be self-complementary if $\overline{G} \cong G$. Evidently, relation (4) holds for all self-complementary graphs. It is worth noting that for self-complementary graphs, $\overline{m} = m$, from which it follows

$$m = \frac{n(n-1)}{4} \tag{5}$$

implying that $n \equiv 0 \,(mod\, 4)$ or $n \equiv 1 \,(mod\, 4)$.

Suppose that the relation (4) holds for some function $g = g(x, y)$. Let α and β be constants, and consider the function $g^* = \alpha\, g(x, y) + \beta$. Then directly from Eq. (2) we see that

$$\sum_{uv \notin \mathbf{E}(G)} \left[\alpha\, g(f_u, f_v) + \beta\right] = \alpha\, \overline{\mathcal{I}}(G) + \beta \overline{m}$$

$$\sum_{uv \notin \mathbf{E}(\overline{G})} \left[\alpha\, g(\overline{f}_u, \overline{f}_v) + \beta\right] = \alpha\, \overline{\mathcal{I}}(\overline{G}) + \beta m.$$

Thus, if $\beta = 0$, then in the case of the function g^*, relation (4) is valid for all graphs G, whereas if $\beta \neq 0$, then (4) holds only for graphs satisfying the condition (5).

If the function $g(x, y)$ is defined so that

$$g(f_u, f_v) = \begin{cases} a & \text{if } \; uv \in \mathbf{E}(G) \\ b & \text{if } \; uv \notin \mathbf{E}(G) \end{cases}$$

then by Eq. (2),

$$\overline{\mathcal{I}}(G) = \overline{m}\, b.$$

Since, in addition

$$g(\overline{f}_u, \overline{f}_v) = \begin{cases} a & \text{if } \; uv \in \mathbf{E}(\overline{G}) \\ b & \text{if } \; uv \notin \mathbf{E}(\overline{G}) \end{cases}$$

we get

$$\overline{\mathcal{I}}(\overline{G}) = m\, b.$$

Therefore, if $b \neq 0$, then the relation (4) will be satisfied by graphs obeying condition (5). If, however, $b = 0$, then the relation (4) will be satisfied by all graphs. Interestingly, the actual value of the parameter a is irrelevant for this conclusion.