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A power penalty method for a 2D fractional partial differential linear complementarity problem governing two-asset American option pricing

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ABSTRACT

In this paper we propose a power penalty method for a linear complementarity problem (LCP) involving a fractional partial differential operator in two spatial dimensions arising in pricing American options on two underlying assets whose prices follow two independent geometric Lévy processes. We first approximate the LCP by a nonlinear 2D fractional partial differential equation (fPDE) with a penalty term. We then prove that the solution to the fPDE converges to that of the LCP in a Sobolev norm at an exponential rate depending on the parameters used in the penalty term. The 2D fPDE is discretized by a 2nd-order finite difference method in space and Crank–Nicolson method in time. Numerical experiments on a model Basket Option pricing problem were performed to demonstrate the convergent rates and the effectiveness of the penalty method.

1. Introduction

Option valuation through a partial differential equation approach has been increasingly attracting much attention from financial engineers, mathematicians and statisticians, ever since the publication of the two seminal papers [4] and [20]. In [4] the authors showed that in a complete market the price of an option on a stock whose price follows a geometric Brownian motion with constant drift and volatility satisfies a second order parabolic partial differential equation, known as the Black–Scholes (BS) equation. However, Gaussian shocks used in BS model often underestimate the probability that stock prices usually exhibit large movements over small time steps which can be demonstrated by empirical financial market data. When jumps are large and rare, a jump-diffusion pricing model can be used to capture them. More details of these models and their numerical solutions can be found in, for example, [1,2,13,30,31]. If there are infinitely many jumps in a finite time interval, an infinite activity Lévy process can be used to capture both frequent small and rare large moves. It has been shown in [6] that, the price of an option on a single asset satisfies a 1D parabolic fractional Black–Scholes (fBS) equation when its underlying asset price follows a geometric Lévy process. This 1D fBS equation and the corresponding American option pricing problem can be solved numerically by the numerical methods proposed recently by us in [7,8]. In [10], Clift and Forsyth proposed an implicit finite difference method for the two dimensional parabolic partial

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integro-differential equation (PIDE) to price two-asset European and American options whose assets follow the correlated finite activity jump diffusion model.

In this work, we shall present a numerical method consisting of a penalty approach and a discretization scheme for pricing American options written on two assets whose prices follow two independent geometric Lévy processes. Under the same assumptions as in [6], it is easy to show that the value of such a two-asset option of European type (eg. Rainbow or Basket Option) is determined by a 2D fBS equation and the value of the corresponding American option is governed by a linear complementarity problem involving the fractional partial differential operator used in the European option model. The latter can also be formulated as a fractional partial differential variational inequality. We comment that, the CGMY jump-diffusion process [5] is also popular in option pricing. An fBS equation for pricing European options has also been developed in [6]. However, in the present work, we only consider the fBS equations and inequalities associated with the geometric Lévy process and will develop algorithms for the fractional differential LCPs based on the CGMY jump-diffusion process in a future paper.

Penalty approaches have been used very successfully for solving constrained optimization problems. In recent years, penalty methods have been used for complementarity or variational inequality problems in both finite and infinite dimensions [3,25,35], particularly those from the valuation of financial options [15,18,19,22,27,33,34,36]. Recently, modern optimization techniques such as the use of grossone theory proposed in [24] in nonlinear programming problems with differentiable penalty functions to determine the penalty parameters has been developed in [11]. In [8], we proposed a power penalty method for solving the fBS equation governing single-asset American option pricing. In this paper, we construct and analyze a power penalty method for the fractional differential complementarity problem arising in pricing the aforementioned two-asset American options. In particular, we will establish a convergence theory for the penalty method proposed. We will then propose a 2nd-order accurate scheme for the discretization of the 2D nonlinear fBS equation in two spatial dimensions generated by the penalty method, based on our recent work in [7] for the 1D fBS equation arising in pricing one-asset options.

While the numerical solution of fractional differential LCPs and fBS equations arising in pricing options written on one risky asset has been discussed in various existing works, to our best knowledge, there are no numerical methods for their 2D counterparts governing the valuation of options on two assets. Therefore, the present work will fill this gap and provide a numerical tool for pricing European and American options of the aforementioned type.

The organization of this paper is as follows. In the next section, we will give a brief account of the fBS equation and fractional differential LCP, along with their initial and boundary conditions, governing the valuation of European and American options written on two independent risky assets. We will also formulate the LCP as a variational inequality and show that the latter problem is uniquely solvable. In Section 3, we will first propose the power penalty method with positive penalty parameters $\lambda > 1$ and k, and consider the unique solvability of the penalty equation. We will then prove that the solution to the penalty equation converges to that of the variational inequality at the rate $\mathcal{O}(\lambda^{-k/2})$. A 2nd-order accurate discretization scheme is proposed in Section 4 for the penalty equation. In Section 5, we will present some numerical experimental results using an American Basket option pricing problem to numerically demonstrate the rates of convergence and usefulness of the numerical method.

2. The option pricing problem

It is shown in [6] that the value of an option whose price follows a geometric Lévy process is governed by a 1D fBS equation. Under the same assumptions as in [6], it is trivial to show that the value U of a European option written on two assets (eg. Rainbow or Basket Option) whose prices S_1 and S_2 follow two independent geometric Lévy processes is determined by the following two-dimensional fBS equation:

$$\mathcal{L}U := -U_t + a_1U_x + a_2U_y - b_1[{}_{-\infty}D_x^{\alpha}U] - b_2[{}_{-\infty}D_y^{\beta}U] + rU = 0 \tag{1a}$$

for $(x, y, t) \in (-\infty, \infty)^2 \times [0, T)$, where $x = \ln S_1$, $y = \ln S_2$, ${}_{-\infty}D_x^{\alpha}U$ and ${}_{-\infty}D_y^{\beta}U$ denote respectively the αth and βth derivatives of U in x and y for α, $\beta \in (1, 2)$, $T > 0$ is the expiry date, $r \geq 0$ is the risk-free rate, $\sigma > 0$ is the volatility of the underlying asset price, and

$$a_1 = -r - \frac{1}{2}\sigma^{\alpha}\sec\left(\frac{\alpha\pi}{2}\right), \quad b_1 = -\frac{1}{2}\sigma^{\alpha}\sec\left(\frac{\alpha\pi}{2}\right) > 0,$$

$$a_2 = -r - \frac{1}{2}\sigma^{\beta}\sec\left(\frac{\beta\pi}{2}\right), \quad b_2 = -\frac{1}{2}\sigma^{\beta}\sec\left(\frac{\beta\pi}{2}\right) > 0.$$

Boundary and terminal conditions can be defined for the above equation depending on the types of options and the strike price K. From the transformations $x = \ln S_1$ and $y = \ln S_2$, we have

$$\lim_{x\to-\infty} U_x = \lim_{x\to-\infty} U_{S_1}e^x = 0, \quad \lim_{y\to-\infty} U_y = \lim_{y\to-\infty} U_{S_2}e^y = 0,$$

since U_{S_1} and U_{S_2} are bounded as $S_1, S_2 \to 0^+$ in practice. In computation, the infinite solution domain $(-\infty, \infty)^2$ has to be truncated into $\Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max})$, where $x_{\min}$, $x_{\max}$, $y_{\min}$ and $y_{\max}$ are four constants satisfying