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Analysis of the method of fundamental solutions for the modified Helmholtz equation

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ABSTRACT

The method of fundamental solutions (MFS) was first proposed by Kupradze in 1963. Since then, the MFS has been extensively applied for solving various kinds of problems in science and engineering. However, few theoretical works have been reported in the literature. In this paper, we devote our work to the error analysis and stability of the MFS for the case of the modified Helmholtz equation. For disk domains, a convergence analysis of the MFS was provided by Li (J. Comput. Appl. Math., 159:113–125, 2004) for solving the modified Helmholtz equation. In this paper, the error bounds of the MFS for bounded and simply connected domains are derived for smooth solutions of the modified Helmholtz equation. The exponential convergence rates can be achieved for analytic solutions. The bounds of condition numbers of the MFS are derived for both disk domains and the bounded and simply connected domains, to give the exponential growth via the number of fundamental solutions used. Numerical experiments are carried out to support the theoretical analysis. Moreover, the analysis of this paper is applied to parabolic equations, and some reviews of proof techniques and the analytical characteristics of the MFS are addressed.

1. Introduction

The method of fundamental solutions (MFS) has been known as a simple and effective boundary meshless method for solving homogeneous problems when the fundamental solution of the given partial differential equation is known. The MFS was originally proposed by Kupradze [20,21] in 1963 but was not well known to the west until the numerical implementation of the MFS was given by Mathon and Johnston in 1977 [34]. Bogomolny [2], Fairweather and Karageorghis [11] and Smyrlis [37] made significant contribution to further develop the MFS. Golberg and Chen [12] extended the MFS to solving nonhomogeneous problems and time-dependent problems through the use of radial basis functions. During the last two decades, we have witnessed a wide spread use of the MFS for solving various types of science and engineering problems. During this period, three major review papers [3,11,12] and monograph of Chen et al. [6], a series of conferences/workshops, and journal special issues have been dedicated to the development of the MFS. Chen et al. [5] report the important progress

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at the first International Workshops on the method of fundamental solutions in Ayia Napa, Cyprus, June 11–13, 2007. In monograph of Kolodziej and Zielinski [19], both the MFS and the method of particular solutions (MPS) in [31] are widely applied for solving many engineering problems, but called differently "*boundary collocation techniques*". There is a large volume of publication of the MFS in terms of numerical computation, but only a relatively small number of papers for theoretical analysis are available in the literature. Our recent efforts are devoted to consolidate the theoretical analysis of the MFS.

The main advantage of the MFS is its simplicity, since the numerical algorithms are straightforward once the fundamental solutions (FS) are known. However, since the source points of the MFS are located outside of the solution domain, an outstanding research topic is how to choose the optimal source location properly. Consequently, there are still a number of challenges to be resolved, such as the difficulty of selecting the fictitious boundary, the ill-conditioning of the resultant matrix, and the lack of theoretical establishment of the MFS. It is the purpose of this paper to focus on the study of the error and stability analyses for the modified Helmholtz equation using the MFS. In Li [23], the MFS was used for solving modified Helmholtz equation, and the convergence error analysis is made only for simple disk domains. In Li [22], the plane wave functions are also proposed for solving modified Helmholtz equation, and the analysis is also confined to disk domains. In this paper, the bounds of errors are derived for the bounded and simply connected domains, to achieve the polynomial convergence rates.

It is known that the resultant matrix of the MFS is very ill-conditioned. Furthermore, the bounds of condition numbers of the MFS are also derived for both disk domains and the bounded and simply connected domains, to achieve the exponential growth rate with respect to the number of fundamental solution used. The exponential growth is strongly connected to the severe ill-conditioning of the MFS. How to handle the ill-conditioning properly is an important issue in numerical implementation; some further techniques on the issue of ill-conditioning will be briefly mentioned in Section 6.

This paper is organized as follows. In Section 2, the MFS is described for the modified Helmholtz equation. In Sections 3 and 4, the error and the stability analyses are explored for the MFS, respectively. In Section 5, numerical experiments are conducted to support the theoretical analysis made. In the last section, applications to parabolic equations are discussed, and some remarks for the MFS are addressed.

2. Description of the method

Let S be a bounded and simply connected domain with the smooth boundary $\Gamma = \partial S$ in 2D. In this paper, we consider the following modified Helmholtz equation

$$(\nabla^2 - \lambda^2)u(x,y) = 0, \qquad (x,y) \in S, \tag{2.1}$$

$$u(x,y) = g(x,y), \quad (x,y) \in \partial S, \tag{2.2}$$

where $\lambda > 0$. Eqs. (2.1) and (2.2) are obtained from the implicit marching schemes from heat or diffusion equations [3]. They can also be derived from the problem of dilute electrolytes, called the Debye–Huckel equation. Eqs. (2.1) and (2.2) are often chosen as the benchmark to develop numerical methods of partial differential equations. In Li [24,32], Eq. (2.1) with $\lambda = 1$ is selected for studying the combined method and the collocation Trefftz method.

Let $P = \rho e^{i\theta}$ and $Q = Re^{i\psi}$ with $i = \sqrt{-1}$, $0 \le \theta, \ \psi \le 2\pi$, and $R > \rho > 0$. Let $r = |PQ| = \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \psi)}$. The fundamental solution of the modified Helmholtz operator in (2.1) is known as follows

$$G(\lambda;\rho,\theta) = \mathcal{K}_0(\lambda r) = \mathcal{K}_0(\lambda\sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \psi)}), \tag{2.3}$$

where $\mathcal{K}_0(\cdot)$ is the Bessel function of the third kind of order zero. Note that the Bessel functions of the third kind of integer order $n \ge 0$ are defined by (see Abramowitz and Stegun [1, p. 253], Tikhonov and Sammariskii [41] and Watson [42])

$$\mathcal{K}_n(x) = \frac{\Gamma(n+1/2)2^n}{x^n\Gamma(1/2)} \int_0^\infty \frac{\cos xu}{(u^2+1)^{n+1/2}} du, \tag{2.4}$$

or

$$\mathcal{K}_n(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh\eta - n\eta} d\eta. \tag{2.5}$$

The approximate solution v_N of v in (2.1) and (2.2) can be written as the linear combination of fundamental solutions G_k

$$v_N = \sum_{k=1}^{N} c_k G_k(\lambda;\rho,\theta), \tag{2.6}$$

where $\{c_k\}$ are the coefficients to be determined, and

$$G_k(\lambda;\rho,\theta) = \mathcal{K}_0(\lambda r_k) = \mathcal{K}_0(\lambda\sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \psi_k)}), \tag{2.7}$$