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Unified treatment of several asymptotic expansions concerning some mathematical constants

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A B S T R A C T

Recently various approximation formulas for some mathematical constants have been investigated and presented by many authors. In this paper, we first find that the relationship between the coefficients p_j and q_j is such that

$$\psi\left(x\sum_{j=0}^{\infty} q_j x^{-j}\right) \sim \ln\left(x\sum_{j=0}^{\infty} p_j x^{-j}\right), \quad x \to \infty,$$

where ψ is the logarithmic derivative of the gamma function (often referred to as psi function) and $p_0 = q_0 = 1$. Next, by using this result, we give a unified treatment of several asymptotic expansions concerning the Euler–Mascheroni constant, Landau and Lebesgue constants, Glaisher–Kinkelin constant, and Choi–Srivastava constants.

1. Introduction

The Euler (or, more precisely, the Euler–Mascheroni) constant γ is defined as follows (see, *e.g.*, [58, Section 1.2]):

$$\begin{aligned}\gamma &:= \lim_{n\to\infty}(H_n - \ln n)\\ &\cong 0.57721\,56649\,01532\,86060\,6512\,09008\,24024\,31042\ldots,\end{aligned} \tag{1.1}$$

where H_n are called the nth harmonic numbers defined by

$$H_n := \sum_{k=1}^{n}\frac{1}{k} \quad (n \in \mathbb{N} := \{1,\ 2,\ 3,\ \ldots\}). \tag{1.2}$$

The constant γ is closely related to the familiar gamma function $\Gamma(z)$ so chosen that $\Gamma(1) = 1$ in the Weierstrass product form of the gamma function (see [1, p. 255, Eq. (6.1.3)]; see also [57, Section 1.1]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right] \quad (|z| < \infty). \tag{1.3}$$

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The logarithmic derivative of the gamma function:

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$$

is known as the psi (or digamma) function. It is well known (see, *e.g.*, [1, p. 258]) that

$$\psi(n+1)=-\gamma+H_n \quad (n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}), \tag{1.4}$$

where the empty sum (as usual, throughout this paper) is understood to be nil.

Young [66] proved that

$$\frac{1}{2(n+1)}<D_n-\gamma<\frac{1}{2n} \quad (n\in\mathbb{N}), \tag{1.5}$$

where

$$D_n:=H_n-\ln n \quad (n\in\mathbb{N}). \tag{1.6}$$

The convergence of the sequence D_n to γ is very slow. By changing the logarithmic term in (1.6), DeTemple [29], Negoi [49] and Chen et al. [19] have presented, respectively, faster asymptotic formulas as follows:

$$H_n-\ln\left(n+\frac{1}{2}\right)=\gamma+O\left(n^{-2}\right) \quad (n\to\infty); \tag{1.7}$$

$$H_n-\ln\left(n+\frac{1}{2}+\frac{1}{24n}\right)=\gamma+O\left(n^{-3}\right) \quad (n\to\infty); \tag{1.8}$$

$$H_n-\ln\left(n+\frac{1}{2}+\frac{1}{24n}-\frac{1}{48n^2}\right)=\gamma+O\left(n^{-4}\right) \quad (n\to\infty). \tag{1.9}$$

Very recently, Chen and Mortici [18] provided a still faster asymptotic formula than those in (1.7)–(1.9):

$$H_n-\ln\left(n+\frac{1}{2}+\frac{1}{24n}-\frac{1}{48n^2}+\frac{23}{5760n^3}\right)=\gamma+O\left(n^{-5}\right) \quad (n\to\infty) \tag{1.10}$$

and posed the following natural question:

Open problem 1.1. For a given $m\in\mathbb{N}$, find the constants p_j $(j=1,2,\ldots,m)$ such that

$$H_n-\ln\left(n\left(1+\sum_{j=1}^{m}\frac{p_j}{n^j}\right)\right) \tag{1.11}$$

is the fastest sequence which would converge to γ.

Yang [65] first presented the solution of the Open problem 1.1 by using Bell polynomials of a logarithmic type. Subsequently, other proofs of the Open problem 1.1 were published by Gavrea and Ivan [34,35], Lin [44], Chen et al. [16]. For example, Lin [44] gave a formula for determining the coefficients of the asymptotic expansion

$$\psi(x+1)\sim\ln\left(x+\frac{1}{2}+\frac{1}{24x}-\frac{1}{48x^2}+\frac{23}{5760x^3}+\frac{17}{3840x^4}-\cdots\right) \quad (x\to\infty), \tag{1.12}$$

and then applied it to give the proof of the Open problem 1.1.

The constants of Landau and Lebesgue are defined, for all $n\in\mathbb{N}_0$, in order, by

$$G_n=\sum_{k=0}^{n}\frac{1}{16^k}\binom{2k}{k}^2 \quad \left(n\in\mathbb{N}_0\right) \tag{1.13}$$

and

$$L_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin\left((n+\frac{1}{2})t\right)}{\sin(\frac{1}{2}t)}\right|\mathrm{d}t \quad (n\in\mathbb{N}_0), \tag{1.14}$$

which play important roles in the theories of complex analysis and Fourier series, respectively.

Recently, the following two open problems were posed in [13,14].

Open problem 1.2. Let h be a given real number. Find the constants $q_j(h)$ $(j\in\mathbb{N})$ such that

$$G_n\sim c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}+\sum_{j=1}^{\infty}\frac{q_j(h)}{(n+h)^j}\right) \quad (n\to\infty),$$