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A common generalization of convolved generalized Fibonacci and Lucas polynomials and its applications

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ABSTRACT

The purpose of this paper is to give a common generalization of convolved generalized Fibonacci and Lucas polynomials and some recurrence relations are established. As applications, we obtain some computational formulas of mixed-multiple sums for $h(x)$-Fibonacci polynomials and Lucas polynomials.

1. Introduction

Fibonacci numbers F_n and Lucas number L_n are defined by

$$F_0=0,\quad F_1=1,\quad F_{n+1}=F_n+F_{n-1},\quad n\geq 1;$$
$$L_0=2,\quad L_1=1,\quad L_{n+1}=L_n+L_{n-1},\quad n\geq 1;$$

respectively. Fibonacci and Lucas numbers and their generalizations have many interesting properties and applications to almost every fields of science and art [14]. In [3], the k-Fibonacci sequences $F_{k,\,n}$ are given by

$$F_{k,0}=0,\quad F_{k,1}=1,\quad F_{k,n+1}=kF_{k,n}+F_{k,n-1},\quad n\geq 1,$$

in studying the recursive application of two geometrical transformations used in the four-triangle longest-edge partition. These numbers have been studied in [2–6,20].

The convolved Fibonacci numbers $F_j^{(r)}$ are defined by

$$(1-t-t^2)^{-r}=\sum_{j=0}^{\infty}F_{j+1}^{(r)}t^j,\quad r\in Z^+,$$

which have been studied in [10,15,16]. The convolved k-Fibonacci numbers have been studied in [18]. In [17], the $h(x)$-Fibonacci and Lucas polynomials are defined by

$$F_{h,0}(x)=0,\quad F_{h,1}(x)=1,\quad F_{h,n+1}(x)=h(x)F_{h,n}(x)+F_{h,n-1}(x),\quad n\geq 1,$$

$$L_{h,0}(x)=2,\quad L_{h,1}(x)=h(x),\quad L_{h,n+1}(x)=h(x)L_{h,n}(x)+L_{h,n-1}(x),\quad n\geq 1;$$

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where $h(x)$ is a polynomial with real coefficients. Their generating functions are

$$\sum_{n=0}^{\infty} F_{h,n+1}(x)t^n = \frac{1}{1-h(x)t-t^2}, \quad \sum_{n=0}^{\infty} F_{h,n}(x)t^n = \frac{t}{1-h(x)t-t^2};$$

$$\sum_{n=0}^{\infty} L_{h,n+1}(x)t^n = \frac{h(x)+2t}{1-h(x)t-t^2}, \quad \sum_{n=0}^{\infty} L_{h,n}(x)t^n = \frac{2-h(x)t}{1-h(x)t-t^2}$$

respectively. Recently, Ramirez [19] introduced the convolved $h(x)$-Fibonacci polynomials $F_{h,n}^{(r)}(x)$ defined by

$$\sum_{n=0}^{\infty} F_{h,n+1}^{(r)}(x)t^n = \frac{1}{(1-h(x)t-t^2)^r},$$

and studied their properties. In fact, the convolved $h(x)$-Lucas polynomials $L_{h,n}^{(r)}(x)$ can be defined by

$$\sum_{n=0}^{\infty} L_{h,n+1}^{(r)}(x)t^n = \frac{(h(x)+2t)^r}{(1-h(x)t-t^2)^r}.$$

Other generalizations for Fibonacci numbers see [1,7,8,11–13,21,22].

The purpose of this paper is to give a common generalization of convolved generalized Fibonacci and Lucas polynomials and some recurrence relations are established. As applications, we obtain some computational formulas of mixed-multiple sums for $h(x)$-Fibonacci polynomials and Lucas polynomials.

2. The extended convolved $h(x)$-Fibonacci–Lucas polynomials

Definition 2.1. Let r and m be positive integers with $r \geq m$. Then the extended convolved $h(x)$-Fibonacci–Lucas polynomials $T_{h,n}^{(r,m)}(x)$ are defined by

$$\sum_{n=0}^{\infty} T_{h,n}^{(r,m)}(x)t^n = \frac{(h(x)+2t)^m}{(1-h(x)t-t^2)^r}. \tag{1}$$

It is easy to see that

$$T_{h,n}^{(r,0)}(x) = F_{h,n+1}^{(r)}(x), \quad T_{h,n}^{(r,r)}(x) = L_{h,n+1}^{(r)}(x),$$

$$T_{h,n}^{(1,0)}(x) = F_{h,n+1}(x), \quad T_{h,n}^{(1,1)}(x) = L_{h,n+1}(x),$$

$$T_{1,n}^{(1,0)}(x) = F_{n+1}, \quad T_{1,n}^{(1,1)}(x) = L_{n+1}.$$

Applying the binomial theorem, we obtain the following expression of $T_{h,n}^{(r,m)}(x)$:

$$T_{h,n}^{(r,m)}(x) = \sum_{k=0}^{\min\{m,n\}} \sum_{i=0}^{[\frac{n-k}{2}]} 2^k \binom{m}{k} \binom{r+n-k-i-1}{n-k-i} \binom{n-k-i}{i} h^{m+n-2k-2i}(x). \tag{2}$$

Taking $n = 0, 1$ in (2), we have

$$T_{h,0}^{(r,m)}(x) = h^m(x), \quad T_{h,1}^{(r,m)}(x) = rh^{m+1}(x) + 2mh^{m-1}(x).$$

Theorem 2.2. *The following identities hold:*

$$T_{h,n}^{(r,m)}(x) = h(x)T_{h,n-1}^{(r,m)}(x) + T_{h,n-2}^{(r,m)}(x) + T_{h,n}^{(r-1,m)}(x), \tag{3}$$

$$T_{h,n}^{(r,m+1)}(x) = h(x)T_{h,n}^{(r,m)}(x) + 2T_{h,n-1}^{(r,m)}(x), \tag{4}$$

$$T_{h,n}^{(r,m)}(x) = \frac{n+1}{r-1}T_{h,n+1}^{(r-1,m-1)}(x) - \frac{2(m-1)}{r-1}T_{h,n}^{(r-1,m-2)}(x). \tag{5}$$

Proof. From identity

$$\frac{(h(x)+2t)^m}{(1-h(x)t-t^2)^r} = \frac{(h(x)+2t)^m(h(x)t+t^2)}{(1-h(x)t-t^2)^r} + \frac{(h(x)+2t)^m}{(1-h(x)t-t^2)^{r-1}},$$