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Degree of approximation for bivariate extension of Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators

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ABSTRACT

In this paper, we introduce the bivariate generalization of the Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators on an unbounded domain and studied the rate of convergence in terms of the Lipschitz class function and complete modulus of continuity. Further, we establish the weighted approximation properties for these operators. The aim of this paper is to obtain the degree of approximation for these bivariate operators in terms of the partial moduli of continuity and the Peetre's K- functional. Then, we give generalization of the operators and investigate their approximations. Furthermore, we show the convergence of the bivariate Chlodowsky-type operators to certain functions by illustrative graphics using Python programming language. Finally, we construct the GBS operators of bivariate Chlodowsky-type q-Bernstein–Stancu–Kantorovich and estimate the rate of convergence for these operators with the help of mixed modulus of smoothness.

1. Introduction

In the last decade, the application of q-Calculus has gained importance significantly in the area of approximation theory [6]. In 1987, Lupas [29] introduced a modification of the Bernstein operators while Philips [33] ten years later introduced another generalization of these operators based on q-integers. The bivariate q-Bernstein polynomials for the Korovkin type approximation are considered and studied in [1,3,30]. In [13,22], Bögel introduced the concept of B-continuous and B-differentiable functions. GBS (Generalized Boolean Sum) operators were introduced by Badea and Cottin in [9]. In the recent years, several researchers contributed to this area of approximation theory [2,4,7–11,13,21,22,25,26,34,35]. The generalization of Kantorovich type operators have been studied in [16–19,27,31,37]. Lately, Karsli and Gupta [28] introduced the q-Bernstein–Chlodowsky operators as a generalization of q-Bernstein operators on an unbounded set as follows:

$$C_n(f, q_n, x)) = \sum_{k=0}^{n} f\left(\frac{[k]_{q_n}}{[n]_{q_n}} a_n\right) \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \left(\frac{x}{a_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q_n^s \frac{x}{a_n}\right),$$

where $0 \le x \le a_n$ and $\{a_n\}$ is a sequence of positive numbers such that $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} \frac{a_n}{[n]_{q_n}} = 0$.

Recently, Vedi and Ozarslan introduced and investigated the Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators [36]. They also obtained the Korovkin type approximation theorem and the rate of convergence of the approximation process

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in terms of the first and the second modulus of continuity and also by means of Lipschitz class functions. For $n_1 \in N$, $p_1 \in N_0$ and $0 \le x \le a_{n_1}$, the Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators are defined in [36] as $C_{n_1+p_1}^{(\alpha_1,\beta_1)}(f; q_{n_1}, x) = \sum_{k_1=0}^{n_1+p_1} \left[{n_1+p_1 \atop k_1}\right]_{q_{n_1}} (\frac{x}{a_{n_1}})^{k_1} (1-\frac{x}{a_{n_1}})_{q_{n_1}}^{n_1+p_1-k_1} \int_0^1 f(\frac{q_{n_1}^{k_1} t_1 + [k_1]_{q_{n_1}} + \alpha_1}{[n_1+1]_{q_{n_1}} + \beta_1} a_{n_1}) d q_{n_1} t_1$ where α_1, $\beta_1 \in R$ with $0 < \alpha_1 \le \beta_1$, $0 < q < 1$.

Agrawal et al. [3] introduced a Bivariate of q-Bernstein–Schurer–Kantorovich operators and investigated their approximation properties. Büyükyazıcı [15] defined the two-dimensional q-Bernstein–Chlodowsky operators as follows:

$$B_{n,m}^{q_n,q_m}(f; x, y) = \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{[k]_{q_n}}{[n]_{q_n}} \alpha_n, \frac{[j]_{q_m}}{[n]_{q_m}} \beta_m \right) \Omega_{k,n,q_n}\left(\frac{x}{\alpha_n}\right) \Omega_{j,m,q_m}\left(\frac{y}{\beta_m}\right)$$

where $x \in [0, \alpha_n], y \in [0, \beta_m]$, and $\Omega_{k,n,q_n}(u) = \left[{n \atop k}\right]_{q_n} u^k \prod_{s=0}^{n-k-1} (1 - q_n^s u)$.

Firstly, let us recall certain notations of q-calculus [32]. For any fixed real number $q > 0$ for each nonnegative integer n, the q-integer $[n]_q$ and q-factorial $[n]_q!$ are defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1 \\ n & \text{if } q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \ldots, [2]_q[1]_q, & \text{if } n \in N \\ 1 & \text{if } n = 0, \end{cases}$$

respectively. Then, for any integers n and k satisfying $0 \le k \le n$, and $q > 0$, the q-binomial coefficient are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The Jackson type q-integral in the interval $[0, a]$ is defined by

$$\int_0^a f(s) d_{q_{n_1}} s = (1 - q_{n_1}) a \sum_{j_1=0}^{\infty} f(a q_{n_1}^{j_1}) q_{n_1}^{j_1}, \ 0 < q_{n_1} < 1,$$

provided that sums converge absolutely.

Suppose that $q_{n_1} > 0, q_{n_2} > 0$, then the Jackson type q-integral for a bivariate function in the interval $[0, a] \times [0, b]$ is given by

$$\int_0^a \int_0^b f(t_1, t_2) d_{q_{n_1}} t_1 d_{q_{n_2}} t_2 = (1 - q_{n_1})(1 - q_{n_2}) ab \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} f(a q_{n_1}^{j_1}, b q_{n_2}^{j_2}) q_{n_1}^{j_1} q_{n_2}^{j_2}, \tag{1}$$

where $q_{n_i} \in (0, 1)$ for $i = 1, 2$.

The structure of the paper is as follows. In Section 2, we introduce the bivariate Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators and study the rate of convergence in terms of the Lipschitz class function and complete modulus of continuity, then we obtain the degree of approximation for these bivariate operators in terms of the partial moduli of continuity and Peetre's K-functional. In Section 3, Korovkin-type theorems are proved. In Section 4, we study the generalization of the Chlodowsky-type generalization of q-Bernstein–Stancu–Kantorovich operators and study their approximation properties. In the last section of the paper, we construct the GBS operators of the Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators defined by (14) and obtain the order of approximation in terms of the mixed modulus of smoothness. Furthermore, we improve the smoothness properties in terms of the mixed K-functional for these GBS operators. In this study, we construct a bivariate case of a new Chlodowsky-type generalization of q-Bernstein–Stancu–Kantorovich operators defined in [36]. In this paper, we define Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators in a rectangular domain $I_{ab} = [0, a] \times [0, b]$ in (2) and investigate their Korovkin type approximation properties. Moreover, we investigate the rate of convergence of the associated GBS operators for the Bögel continuous and Bögel differentiable functions.

2. The construction of the operators

Let $\{q_{n_1}\}$ and $\{q_{n_2}\}$ be the sequences of real numbers such that $0 < q_{n_i} < 1$, and $\lim_{n_i \to \infty} q_{n_i} = 1$ for $i = 1, 2$, and $\{a_{n_1}\}$ and $\{b_{n_2}\}$ be sequences of positive real numbers that satisfy the following properties:

$$\lim_{n_1 \to \infty} a_{n_1} = \lim_{n_2 \to \infty} b_{n_2} = \infty, \text{ and } \lim_{n_1 \to \infty} \frac{a_{n_1}}{[n]_{q_{n_1}}} = \lim_{n_2 \to \infty} \frac{b_{n_2}}{[n_2]_{q_{n_2}}} = 0.$$

Further, let $\delta_{n_1}(x) = C_{n_1+p_1}^{(\alpha_1,\beta_1)}((u-x)^2, q_{n_1}, x)$ and $\delta_{n_2}(y) = C_{n_2+p_2}^{(\alpha_2,\beta_2)}(v-y)^2, q_{n_2}, y)$. Next, let us define the real parameters α_1, β_1, α_2 and β_2 satisfying the condition $0 \le \alpha_1 \le \beta_1, 0 \le \alpha_2 \le \beta_2$. If $f \in C(I_{a_{n_1} b_{n_2}})$ and $0 < q_{n_1}, q_{n_2} < 1$, then, we construct