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The center and cyclicity problems for some analytic maps[☆]

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ABSTRACT

The center variety and bifurcations of limit cycles from the center for maps $f(x) = -\sum_{k=0}^{\infty} a_k x^{k+1}$ arising from

$$x + y + \sum_{j=0}^{n} \alpha_{n-j,j} x^{n-j} y^j = 0$$

are considered. Motivated by a general result for $n = 2\ell + 1$ we investigate the center and cyclicity problem for n being even. We review results for $n = 2$ and $n = 4$ and perform the analysis for $n = 6, 8, 10$. Finally, we state some conjectures for general $n = 2\ell$.

1. Introduction

The research of local dynamics of maps

$$f(x) = -\sum_{k=0}^{\infty} a_k x^{k+1} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ represents an irreducible branch of algebraic curves given by

$$x + y + \sum_{i+j=2}^{n} \alpha_{ij} x^i y^j = 0 \tag{1.2}$$

was considered in [8–10,12]. In (1.1) f is assumed to be a real function of one real variable (i.e. $x \in \mathbb{R},\ a_k \in \mathbb{R}$). If (1.2) is solved for y and regarding the solution as $f(x)$, one may consider (1.1) as a discrete dynamical system with a nontrivial (nonhyperbolic) singular point $x = 0$. The case when (1.2) is of the form

$$x + y + \sum_{j=0}^{n} \alpha_{n-j,j} x^{n-j} y^j = 0 \tag{1.3}$$

is called a homogeneous case (of degree n). For map (1.1) we denote by f^p ($p \in \mathbb{N}$) the pth iterate of the map. The singular point $x = 0$ of the map (1.1) is called

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1. *a stable focus*, if there exists an $\varepsilon > 0$ such that for all x for which $|x| < \varepsilon$ we have $\lim_{k\to\infty} f^k(x) = 0$,
2. *an unstable focus*, if it is a stable focus for the (inverse) map f^{-1},
3. *a center*, if there exists an $\varepsilon > 0$ such that for all x for which $|z| < \varepsilon$ the equality $f^2(x) = x$ holds.

A point $x_0 > 0$ is called a *limit cycle* of the map (1.1) if x_0 is an *isolated root* of the equation

$$f^2(x) - x = 0.$$

The cyclicity of centers in (1.1) arrising from (1.3) for n being odd is completely resolved in [10]. Some results for different maps arising from (1.2) and (1.3) were obtained in [8], where maps of the form (1.3) with homogeneous polynomials of degree 2 and 4 are considered. The main results of this paper are solutions to the center and cyclicity problems for (2.1) with $n = 6, 8$ and 10.

The study of the center conditions gives rise to the Lyapunov constants and the study of cyclicity of periodic points of (1.1) in a neighborhood of $x = 0$ is similar to the study of cyclicity in case of a class of planar polynomial system of the form

$$\begin{aligned} \dot{x} &= -y + P(x, y), \\ \dot{y} &= x + Q(x, y), \end{aligned} \tag{1.4}$$

where P and Q are polynomials beginning with quadratic terms. Roughly speaking the cyclicity problem can be considered by examining the coefficients and isolated zeros of the (Poincaré) return map associated to the vector field corresponding to (1.4). The analogy between (1.1) and (1.4) was noticed already in [12] and used in [1,9,10,12] and recently in [8]. In [8] the approach described by Christopher [1] was adopted to discrete dynamical systems (1.1) originating from (1.2).

Analogously to (1.4) one can consider the stable unstable foci and centers for maps (1.1). The analogy is based on the return and difference maps yielding the Lyapunov coefficients similar as in the continuous case (1.4). This is described in detail in next section together with other preliminary results needed for our study in this paper. Thus, in the following section also the technique for estimating the cyclicity of centers in maps (1.1) adopted in [8] (originally stated for continuous systems [1,4,7]) is summarized. The main results (estimating the cyclicity for $n = 6, 8, 10$) are presented in Section 3, where some conjectures for $n = 2\ell,\ \ell \in \mathbb{N}$, based on results for maps (1.3) with $n = 2, 4, 6, 8$ and 10 are stated. Finally, in the last section some conclusions are presented.

2. Preliminary results for center and cyclicity problems of maps (1.1)

In this section we summarize some important definitions and results concerning the dynamics of maps given by (1.1) including the main theorem from Mencinger et al. [8]. For more details see [9,10] and in particular [8].

Consider the Eq. (1.2), where α_{ij}, x, y are real, which has a unique analytic solution of the form (1.1). From the study of solutions to (1.2) we have the following problem: how to find in the space of coefficients $\{\alpha_{ij}\}$ the manifold on which the map f has a center at the origin and to investigate the bifurcation of limit cycles of such map.

In order to study the center problem we use the approach based on Lyapunov function, Φ, as done in [9,12]. If Φ is a Lyapunov function of the map (1.1) then

$$\Phi(f(x)) - \Phi(x) = g_2x^4 + g_4x^6 + \cdots + g_{2k}x^{2k+2} + \cdots, \tag{2.1}$$

and the coefficients g_{2k} from (2.1) are called *focus quantities*. Moreover, its formal series is given by

$$\Phi(x) = x^2\left(1 + \sum_{k=1}^{\infty} b_kx^k\right). \tag{2.2}$$

As in [8–10] in practice we use the truncated approximation of f

$$f_N(x) = -x - \sum_{k=1}^{N} a_kx^{k+1}$$

and the truncated Lyapunov function Φ_N, defined by $\Phi(x) = x^2\left(1 + \sum_{k=1}^{N} b_kx^k\right)$ to obtain focus quantities and the center variety.

The authors in [9] proved that if $g_{2k} = 0$ for all $k \in \mathbb{N}$ then the map (1.1) has a center at the origin. If $g_2 = \cdots = g_{2k-2} = 0$ and $g_{2k} \neq 0$ then $x = 0$ is a stable (respectively, unstable) focus when $g_{2k} < 0$ (respectively, $g_{2k} > 0$) for map (1.1). This actually confirms that the concept of Lyapunov function (2.2) for (1.1) is analogous to the concept of Lyapunov function for planar systems of ODEs and we can define the Poincaré return map

$$\mathrm{R}(x) = f^2(x) = x + c_2x^3 + c_3x^4 + \cdots$$

and the difference map

$$\mathrm{P}(x) = f^2(x) - x = c_2x^3 + c_3x^4 + \cdots. \tag{2.3}$$

By the definition, a limit cycle is a (positive) isolated root of difference function (2.3).