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Computing the k-metric dimension of graphs

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ABSTRACT

Given a connected graph $G=(V,E)$, a set $S\subseteq V$ is a k-metric generator for G if for any two different vertices $u,v\in V$, there exist at least k vertices $w_1,\dots,w_k\in S$ such that $d_G(u,w_i)\neq d_G(v,w_i)$ for every $i\in\{1,\dots,k\}$. A metric generator of minimum cardinality is called a k-metric basis and its cardinality the k-metric dimension of G. We make a study concerning the complexity of some k-metric dimension problems. For instance, we show that the problem of computing the k-metric dimension of graphs is NP-hard. However, the problem is solved in linear time for the particular case of trees.

1. Introduction

Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative real numbers. A *metric space* is a pair (X, d), where X is a set of points and $d: X\times X\rightarrow \mathbb{R}_{\geq 0}$ satisfies $d(x,y)=0$ if and only if $x=y$, $d(x,y)=d(y,x)$ for all $x, y\in X$ and $d(x,y)\leq d(x,z)+d(z,y)$ for all $x, y, z\in X$. A *generator* of a metric space (X, d) is a set S of points in the space with the property that every point of the space is uniquely determined by the distances from the elements of S. A point $v\in X$ is said to *distinguish* two points x and y of X if $d(v,x)\neq d(v,y)$. Hence, S is a generator if and only if any pair of points of X is distinguished by some element of S.

Let $\mathbb{N}$ denote the set of nonnegative integers. Given a connected graph $G=(V,E)$, we consider the function $d_G: V\times V\rightarrow \mathbb{N}$, where $d_G(u,v)$ is the length of a shortest path between u and v. Clearly, (V, d_G) is a metric space. The diameter of a graph is defined with this metric.

A vertex set $S\subseteq V$ is said to be a *metric generator* for G if it is a generator of the metric space (V, d_G). A minimum metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$. Motivated by some problems regarding unique location of intruders in a network, the concept of metric dimension of a graph was introduced by Slater [28], where the metric generators were called *locating sets*. The concept of metric dimension of a graph was also introduced by Harary and Melter [17], where metric generators were called *resolving sets*. Applications of this invariant to the navigation of robots in networks are discussed in [21] and applications to chemistry in [19,20]. This graph parameter was studied further in a number of other papers including, for instance [3,6,7,22,23,30]. Several variations of metric generators

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including resolving dominating sets [5], locating dominating sets [25], independent resolving sets [8], local metric sets [24], strong resolving sets [27], etc. have been introduced and studied.

On the other hand, complexity studies concerning the metric dimension of graphs have recently attracted the attention of several researchers. This is mainly based on the fact that finding the metric dimension of graphs is NP-hard [21], even when restricted to planar graphs [9]. However, there exist a linear-time and a polynomial-time algorithm for determining the metric dimension for trees [21] and outerplanar graphs [9], respectively. For these reasons, many efforts have been made to computationally solve the problem of finding a metric generator of a graph in the last few years. For instance, an increasing interest into algorithmic questions on this topic has been raised (see [10,11,15,18] as some examples).

From now on we consider an extension of the concept of metric generators introduced in [12] by the authors of this paper, and also independently by Adar and Epstein in [1]. Given a simple and connected graph $G=(V,E)$, a set $S\subseteq V$ is said to be a *k-metric generator* for G if and only if any pair of vertices of G is distinguished by at least k elements of S, *i.e.*, for any pair of different vertices $u,v\in V$, there exist at least k vertices $w_1,w_2,\dots,w_k\in S$ such that

$$d_G(u,w_i)\neq d_G(v,w_i),\ \text{for every } i\in\{1,\dots,k\}. \tag{1}$$

A k-metric generator of minimum cardinality in G is called a *k-metric basis* and its cardinality the *k-metric dimension* of G, which is denoted by $\dim_k(G)$, [12]. Note that every k-metric generator S satisfies that $|S|\geq k$ and, if $k>1$, then S is also a $(k-1)$-metric generator. Moreover, 1-metric generators are the standard metric generators (resolving sets or locating sets as defined in [17] or [28], respectively). Notice that if $k=1$, then the problem of checking if a set S is a metric generator reduces to check condition (1) only for those vertices $u,v\in V\setminus S$, as every vertex in S is distinguished at least by itself. Also, if $k=2$, then condition (1) must be checked only for those pairs having at most one vertex in S, since two vertices of S are distinguished at least by themselves. Nevertheless, if $k\geq 3$, then condition (1) must be checked for every pair of different vertices of the graph. The k-metric dimension of connected graphs has been also studied in [2,4,13,14]. Among them, a remarkable study concerns an interesting application of the k-metric generators (whether $k\geq 3$) to the theory of error correcting codes which was presented in [4].

In this article we show the NP-hardness of the problem of computing the k-metric dimension of graphs. To do so, we first prove that the decision problem regarding whether $\dim_k(G)\leq r$ for some graph G and some integer $r\geq k+1$ is NP-complete. The particular case of trees is separately addressed, based on the fact that for trees, the problem mentioned above becomes polynomial. We must remark that, for the tree graphs, a similar approach was also dealt with in [2] for the case $k=2$. We say that a connected graph G is *k-metric dimensional* if k is the largest integer such that there exists a k-metric basis for G. We also show that the problem of finding the integer k such that a graph G is k-metric dimensional can be solved in polynomial time. The reader is referred to [12] for combinatorial results on the k-metric dimension, including tight bounds and some closed formulae. The article is organized as follows. In Section 2 we analyze the problem of computing the largest integer k such that there exists a k-metric basis. In Section 3 we show that the decision problem regarding whether the k-metric dimension of a graph does not exceed a positive integer is NP-complete, which gives also the NP-hardness of computing $\dim_k(G)$ for any graph G. The procedure of such proof is done by using some similar techniques like those ones already presented in [21] while studying the computational complexity problems related to the standard metric dimension of graphs. Finally, in Section 4 we give an algorithm for determining the value of k such that a tree is k-metric dimensional and present two algorithms for computing the k-metric dimension and obtaining a k-metric basis of any tree. We also show that all algorithms presented in this section run in linear time.

Throughout the paper, we use the notation $K_{1,n-1}$, C_n and P_n for star graphs, cycle graphs and path graphs of order n, respectively. For a vertex v of a graph G, $N_G(v)$ denotes the set of neighbors or *open neighborhood* of v in G. The *closed neighborhood*, denoted by $N_G[v]$, equals $N_G(v)\cup\{v\}$. If there is no ambiguity, we simple write $N(v)$ or $N[v]$. We also refer to the degree of v as $\delta(v)=|N(v)|$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

2. *k*-metric dimensional graphs

Notice that if a graph G is a k-metric dimensional graph, then for every positive integer $k'\leq k$, G has at least a k'-metric basis. Since for every pair of vertices x, y of a graph G we have that they are distinguished at least by themselves, it follows that the whole vertex set $V(G)$ is a 2-metric generator for G and, as a consequence it follows that every graph G is k-metric dimensional for some $k\geq 2$. On the other hand, for any connected graph G of order $n>2$ there exists at least one vertex $v\in V(G)$ such that $\delta(v)\geq 2$. Since v does not distinguish any pair $x,y\in N_G(v)$, there is no n-metric dimensional graph of order $n>2$.

Remark 1 [12]**.** Let G be a k-metric dimensional graph of order n. If $n\geq 3$ then, $2\leq k\leq n-1$. Moreover, G is n-metric dimensional if and only if $G\cong K_2$.

Next we present a characterization of k-metric dimensional graphs already known from [12]. To this end, we need some additional terminology. Given two vertices x, $y\in V(G)$, we say that the set of *distinctive vertices* of x, y is

$$\mathcal{D}_G(x,y)=\{z\in V(G): d_G(x,z)\neq d_G(y,z)\}.$$

Theorem 2 [12]**.** *A graph G is k-metric dimensional if and only if* $k=\min_{x,y\in V(G)}|\mathcal{D}_G(x,y)|$.