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Radiative transfer with delta-Eddington-type phase functions

Weimin Han [a,*], Feixiao Long [b], Wenxiang Cong [b], Xavier Intes [b], Ge Wang [b]

[a] *Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA*
[b] *Biomedical Imaging Center, Department of Biomedical Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180, USA*

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ABSTRACT

The radiative transfer equation (RTE) arises in a wide variety of applications, in particular, in biomedical imaging applications associated with the propagation of light through the biological tissue. However, highly forward-peaked scattering feature in a biological medium makes it very challenging to numerically solve the RTE problem accurately. One idea to overcome the difficulty associated with the highly forward-peaked scattering is through the use of a delta-Eddington phase function. This paper is devoted to an RTE framework with a family of delta-Eddington-type phase functions. Significance in biomedical imaging applications of the RTE with delta-Eddington-type phase functions are explained. Mathematical studies of the problems include solution existence, uniqueness, and continuous dependence on the problem data: the inflow boundary value, the source function, the absorption coefficient, and the scattering coefficient. Numerical results are presented to show that employing a delta-Eddington-type phase function with properly chosen parameters provides accurate simulation results for light propagation within highly forward-peaked scattering media.

1. Introduction

The radiative transfer equation (RTE) arises in a wide variety of applications, such as astrophysics [22], atmosphere and ocean [26,31], heat transfer [20], neutron transport [7,9], optical molecular imaging [21,28], and so on. Recently, there is much interest in analysis and numerical simulation of the RTE and its related inverse problems, motivated by applications in biomedical optics [2,3,5,11–15,23,25].

Photon propagation in biological or engineered tissues can be well described by the radiative transport equation (RTE). However, the direct solution of the RTE is computationally expensive because of the dimensionality of the equation and the complexity of the phase function. It is rather common in practice that the diffusion approximation is based upon to enable optical molecular tomographic techniques that reveal optically labeled molecular and cellular activities in vivo. A majority of such studies target small animal models of human diseases and 3D tissue engineering constructs of regenerative functionalities. Photon propagation in these media is strongly affected by scattering. When samples are not large, characteristic forward scattering is observable and responsible for substantial components in the measurement. Inspired by this observation, delta-Eddington-type phase functions were proposed to model the underlying physics, simplify the solution of the RTE, and successfully applied in multiple applications. Therefore, it is desirable and timely to generalize this approach and establish its theoretical foundation.

* Corresponding author.
E-mail address: weimin-han@uiowa.edu (W. Han).

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The RTE for the complicated process of absorption and scattering of light within the biological medium is

$$\omega\cdot\nabla u + \mu_t u - \mu_s S u = f \quad \text{in } X \times \Omega. \tag{1.1}$$

Here, X is a domain in $\mathbb{R}^3$ occupied by the biological medium and Ω is the unit sphere in $\mathbb{R}^3$ for the directions of the photon propagation. A generic point in X is denoted by $\boldsymbol{x}$ whereas a generic point in Ω is denoted by $\boldsymbol{\omega}$. The symbol ∇ stands for the gradient with respect to the spatial variable $\boldsymbol{x}$. The unknown function $u(\boldsymbol{x}, \boldsymbol{\omega})$ is the angular flux at the point $\boldsymbol{x}$ in the direction $\boldsymbol{\omega}$. The RTE (1.1) contains two medium parameters, the total cross-section $\mu_t(\boldsymbol{x})$ and the scattering cross-section $\mu_s(\boldsymbol{x})$, that are related by $\mu_t = \mu_a + \mu_s$ with μ_a being the absorption cross-section. The integral operator S is given by the formula

$$Su(x,\omega) = \int_\Omega p(\hat{\omega}\cdot\omega)\, u(x,\hat{\omega})\, d\hat{\omega}, \tag{1.2}$$

where the phase function $p(\hat{\omega}\cdot\omega)$ is non-negative and is normalized:

$$\int_\Omega p(\hat{\omega}\cdot\omega)\, d\hat{\omega} = 1,$$

or equivalently,

$$\int_{-1}^{1} p(t)\, dt = \frac{1}{2\pi}.$$

The function $f(\boldsymbol{x}, \boldsymbol{\omega})$ represents a source density.

The phase function p describes the scattering property of the biological medium. The precise form of the phase function is usually unknown for applications, and a benchmark choice is the Henyey–Greenstein phase function (cf. [16]):

$$p_{HG,g}(t) = \frac{1-g^2}{4\pi\left(1+g^2-2gt\right)^{3/2}}, \quad t := \hat{\omega}\cdot\omega \in [-1,1],$$

where the parameter $g \in (-1,1)$ is the anisotropy factor of the scattering medium. For isotropic scattering, $g = 0$; for forward scattering, $g > 0$; and for backward scattering, $g < 0$. For applications in biomedical imaging, the value of g is typically between 0.9 and 0.95. For this range of the value of g, the corresponding integral operator (1.2) presents numerical singularity, bringing in additional difficulty in numerically solving the RTE problem. The biological tissue scatters light strongly in the forward direction, and so it is natural to approximately model the effect of the strongly forward scattering through the inclusion of a delta function in the phase function. In this paper, we consider the RTE problem with a general delta-Eddington-type phase function of the following form

$$p(t) = \frac{1}{4\pi}[(1-p_0)\, r(t) + 2\, p_0 \delta(1-t)], \tag{1.3}$$

where $p_0 \in [-1,1]$ is the weighting factor measuring the anisotropy of the photon scattering, δ is the Dirac delta function, and $r(t)$ represents a remainder part of the phase function which is smooth and slowly varying. For strongly forward peaked media, p_0 is less than but close to 1:

$$1-\varepsilon < p_0 < 1,$$

where $\varepsilon > 0$ is a small number. Physical considerations dictate that the remainder function $r(t)$ satisfies the following condition:

$$r(t) \ge 0, \quad \frac{1}{2}\int_{-1}^{1} r(t)\, dt = 1. \tag{1.4}$$

The formula (1.3) includes as particular cases several phase functions proposed in the literature. We list some of them in the following.

The transport approximation [10] corresponds to the choice $r(t) = 1$, i.e., the phase function is the sum of a forward delta function and an isotropic scattering function.

The delta-Eddington phase function [17]

$$p_{dE}(t) = \frac{1}{4\pi}\left[(1-p_0)\left(1+3g't\right) + 2\, p_0\delta(1-t)\right], \tag{1.5}$$

corresponds to the choice

$$r(t) = 1 + 3g't, \tag{1.6}$$

where g' is an asymmetry factor of the phase function used to modulate the weakly anisotropic scattering. The phase function (1.5) is a linear combination of a forward delta function and a weakly anisotropic scattering function. Formally, the transport approximation is a special case of the delta-Eddington phase function with $g' = 0$. Note that the condition (1.4) reduces to

$$-\frac{1}{3} \le g' \le \frac{1}{3}.$$