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On construction of multivariate Parseval wavelet frames ☆

M. Skopina

Saint Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg 199034, Russia

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ABSTRACT

A new method for the construction of compactly supported Parseval wavelet frames in $L_2(\mathbb{R}^d)$ with any preassigned approximation order n for arbitrary matrix dilation M is proposed. The number of wavelet functions generating a frame constructed in this way is less or equal to $(d+1)|\det M| - d$. The method is algorithmic, and the algorithm is simple to use. The number of generating wavelet functions can be reduced to $|\det M|$ for a large class of matrices M.

1. Introduction

Multivariate Parseval (tight) wavelet frames and orthonormal wavelet bases are very useful for applications in many areas. Compactly supported wavelets are of a special interest to engineers. Also it is desirable to have wavelet frames/bases with good approximation properties for applications. A general scheme for construction of compactly supported wavelet bases and frames is well known [23,30], however there are serious problems with its implementation in the multidimensional case. The construction of wavelet bases rests on the following open algebraic problem: is it possible to extend any suitable row to a unitary matrix whose entries are the trigonometric polynomials? There is a conjecture that the answer is negative. This difficulty can be easily avoided in the construction of frames by increasing the redundancy, but there is another difficulty connected with the absence of an analog of the Riesz lemma in the multidimensional case. A generalized analog of Riesz lemma holds in the two-dimensional case. Namely, Scheiderer [25] proved that any nonnegative trigonometric polynomial of two variables can be represented as a finite sum of squared magnitudes of trigonometric polynomials (which is also called Hermitian sum of squares decomposition). Unfortunately, a similar statement does not hold if $d > 2$, which was recently established by Charina et al. [3]. The same authors proved in [4] that the statement is valid for the trigonometric polynomials with positive coefficients. Dritschel [9] proved that the statement is true for strictly positive trigonometric polynomials. However absence of this statement in the general case is a serious obstacle for frame construction. Different approaches for the construction of multivariate compactly supported Parseval wavelet frames have been proposed in many papers, see [1,2,4–8,13–15,19,27,28] and references therein. In particular, a method developed by Han [13,14] is based on the factorization of the dilation matrix M in the form $M = EDF$ with all E, D, F being integer matrices such that $|det(E)| = |det(F)| = 1$ and D is a diagonal matrix. To construct Parseval M-wavelet frames, one needs to construct Parseval D-wavelet frames, from which one can simply use tensor product. A generalization of this approach was recently suggested in [15], where a so-called projection method was suggested. This method gives a way for constructing functions and masks by integrating multidimensional functions along parallel superplanes. We are interested in algorithmic methods leading to the construction of compactly supported Parseval frames providing an arbitrary large approximation order. The

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E-mail addresses: skopina@MS1167.spb.edu, skopinam@mail.ru

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number of generators of such frames is desirable to be as small as possible. Theoretically, the minimal possible number of generators of a Parseval frame with matrix dilation M is $|\det M|-1$. For $M=2I_d$ such frames with minimal number of generators can be easily constructed using Daubechies orthogonal masks. The problem is much more complicated for arbitrary M.

Existence of compactly supported Parseval wavelet frames in $L_2(\mathbb{R}^d)$ with any preassigned approximation order n and the corresponding smoothness for an arbitrary matrix dilation M was proved by Han [13]. Moreover, he proved that such frames are generated by less than $\left(\frac{3}{2}\right)^d|\det M|$ wavelet functions. In the present paper, we provide an algorithmic method for the construction of Parseval wavelet frames with any preassigned approximation order n and such that the number of generating wavelet functions does not exceed $(d+1)|\det M|-d$. Our method is based on the polyphase approach developed in [26,27]. Our algorithm is very simple to use and explicit. Computations are needed only to find several Hermitian square roots of trigonometric polynomials of one variable. The number of generating wavelet functions can be reduced for a large enough class of matrices M. Namely, if all entries of some column of M are divisible by $|\det M|$, then the algorithm can be simplified, and the number of wavelet functions does not exceed $|\det M|$ in this case. For the same class of matrices, using an orthogonal mask of one variable with respect to the dilation factor $|\det M|$, we give explicit formulas for masks generating frames with minimal number of generators. We do not guarantee smoothness (regularity) of refinable and wavelet functions constructed by our method. Although necessary condition for smoothness of order n is satisfied, sufficient condition should be checked in each individual case.

2. Notation and basic facts

We use the following notation: $\mathbb{N}$ is the set of positive integers, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, $x=(x_1,\dots,x_d)$, $y=(y_1,\dots,y_d)$ are its elements (vectors), $(x,y)=x_1y_1+\dots+x_dy_d$, $|x|=\sqrt{(x,x)}$, $\mathbf{0}=(0,\dots,0)\in\mathbb{R}^d$, $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$, I_d denotes the unit $d\times d$ matrix. A^* is the conjugate matrix to A.

For $x,y\in\mathbb{R}^d$, the relation $x>y$ means $x_j>y_j$, $j=1,\dots,d$; $\mathbb{Z}_+^d=\{x\in Z^d:\ x\geq\mathbf{0}\}$.

For $\alpha,\beta\in\mathbb{Z}_+^d$ and $a,b\in\mathbb{R}^d$, we set

$$\alpha!=\prod_{j=1}^d\alpha_j!,\quad \binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!},\quad a^b=\prod_{j=1}^d {a_j}^{b_j},\quad [\alpha]=\sum_{j=1}^d\alpha_j,\quad D^\alpha f=\frac{\partial^{[\alpha]}f}{\partial^{\alpha_1}x_1,\dots,\partial^{\alpha_d}x_d},$$

where δ_{ab} denotes the Kronecker delta.

Let M be a non-degenerate $d\times d$ integer matrix whose eigenvalues are bigger than 1 in module (matrix dilation). We say that numbers $k,n\in\mathbb{Z}^d$ are congruent modulo M (write $k\equiv n \pmod M$) if $k-n=M\ell$, $\ell\in\mathbb{Z}^d$. The integer lattice $\mathbb{Z}^d$ is split into cosets with respect to the introduced relation of congruence. The number of cosets is equal to $|\det M|$ (see, e.g., [21, Section 2.2]). Choose an arbitrary representative from each coset, call them digits and denote the set of digits by $D(M)$. In the sequel, we consider that such a matrix M is fixed, $m=|\det M|$, $D(M)=\{s_0,\dots,s_{m-1}\}$, $s_0=\mathbf{0}$.

If f is a function defined on $\mathbb{R}^d$, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^d$, then

$$f_{jk}(x):=m^{j/2}f(M^jx+k),\quad x\in\mathbb{R}^d.$$

The Fourier transform of a function $f\in L_2(\mathbb{R}^d)$ will be denoted by $\widehat{f}$.

For any trigonometric polynomial m_ν of d variables, there exists a unique set of trigonometric polynomials $\mu_{\nu k}$, $k=0,\dots,m-1$, (polyphase components of m_ν) such that

$$m_\nu(x)=\frac{1}{\sqrt{m}}\sum_{k=0}^{m-1}e^{2\pi i(s_k,x)}\mu_{\nu k}(M^*x) \tag{1}$$

(see., e.g., [21, Section 2.6]).

A wavelet system $\{\psi_{jk}^{(\nu)}\}$, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^d$, $\nu-1,\dots,r$, where $\psi^{(\nu)}\in L_2(\mathbb{R}^d)$, is said to be a Parseval frame (the notion of frame was introduced by Duffin and Schaeffer [10]) if the Parseval equality

$$\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^d}\sum_{\nu=1}^{r}|\langle f,\psi_{jk}^{(\nu)}\rangle|^2=\|f\|^2$$

holds for all $f\in L_2(\mathbb{R}^d)$. It is well known that the Parseval frames provide the orthogonal-type expansions

$$f=\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^d}\sum_{\nu=1}^{r}\langle f,\psi_{jk}^{(\nu)}\rangle\psi_{jk}^{(\nu)} \tag{2}$$

for all $f\in L_2(\mathbb{R}^d)$.

A general scheme for the construction of compactly supported Parseval wavelet frames based on so-called *Unitary Extension Principle* was developed by Ron and Shen [23], One starts with a function $\varphi\in L_2(\mathbb{R}^d)$ satisfying a refinable equation

$$\widehat{\varphi}(x)=m_0(M^{*-1}x)\widehat{\varphi}(M^{*-1}x), \tag{3}$$