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A fast algorithm for the inversion of Abel's transform

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ABSTRACT

We present a new algorithm for the computation of the inverse Abel transform, a problem which emerges in many areas of physics and engineering. We prove that the Legendre coefficients of a given function coincide with the Fourier coefficients of a suitable periodic function associated with its Abel transform. This allows us to compute the Legendre coefficients of the inverse Abel transform in an easy, fast and accurate way by means of a single Fast Fourier Transform. The algorithm is thus appropriate also for the inversion of Abel integrals given in terms of samples representing noisy measurements. Rigorous stability estimates are proved and the accuracy of the algorithm is illustrated also by some numerical experiments.

1. Introduction

The subject of this paper is the analysis and the numerical solution of the Abel integral equation of the first kind:

$$g(x) = (Af)(x) \doteq \int_0^x \frac{f(y)}{\sqrt{x-y}}\,\mathrm{d}y \qquad (0 \leqslant x \leqslant 1). \tag{1}$$

In (1), $g(x) \in L^2(0,1)$ represents the known data function, and $f(x) \in L^2(0,1)$ is the unknown function to be computed. We can assume, with no loss of generality, $g(0) = 0$. Therefore, Eq. (1) defines a linear compact operator A: $L^2(0,1) \to L^2(0,1)$ [23]. Abel's integral equation plays an important role in many areas of science. Its most extensive use is for the determination of the radial distribution of cylindrically symmetric physical quantities, e.g. the plasma emission coefficients, from line-of-sight integration measurements. In X-ray tomography, the object being analyzed is illuminated by parallel X-ray beams and an Abel equation of type (1) relates the intensity profile of the transmitted rays (the data function g) to the object's radial density profile (the unknown function f) [2,9]. Abel inversion is widely used in plasma physics to obtain the electronic density from phase-shift maps obtained by laser interferometry [35] or radial emission patterns from observed plasma radiances [21,32]. Photoion and photoelectron imaging in molecular dynamics [17], evaluation of mass density and velocity laws of stellar winds in astrophysics [13,30], and atmospheric radio occultation signal analysis [28,37] are additional fields which frequently require the numerical solution of Abel's equations of type (1).

The exact solution to (1) traces back to Abel's memoir [1] (see also [12]):

$$f(y) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{g(x)}{\sqrt{y-x}}\,\mathrm{d}x = \frac{1}{\pi}\int_0^y \frac{g'(x)}{\sqrt{y-x}}\,\mathrm{d}x \qquad (\text{with } g(0) = 0), \tag{2}$$

and existence results are conveniently given in Ref. [23] for pairs of functions f and g belonging to a variety of functional spaces (e.g., Hölder spaces and Lebesgue spaces).

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Actual difficulties arise when the Abel inversion has to be computed from input data which are noisy and finite in number, as when the data represent experimental measurements. In this case Eq. (2) is often of little practical utility since it requires the numerical differentiation which tends to amplify the errors. The Abel inversion is in fact a (mildly) ill-posed problem since the solution does not depend continuously on the data: slight inaccuracies in the input data may lead to a solution very *far* from the true one. Stated in other words, since the operator A: $L^2(0, 1) \to L^2(0, 1)$ is compact, its inverse A^{-1} cannot be continuous in the L^2-norm [23]. It is therefore crucial to set up numerical algorithms which yield a stable solution to problem (1). For these reasons, various numerical methods for the inversion of Abel's operator have been proposed. In Refs. [16,21,27] input data are represented through cubic spline and then the inverse Abel transformation is applied to get the solution. Iterative schemes [38] have been proved to be rather stable but are time-consuming. Inversion techniques based on the Fourier-Hankel transform are discussed in Refs. [5,29]. Numerical methods developed within regularization schemes suitable for problem (1) have also been presented [8,24]. The representation of input data and solution in various orthonormal basis in Hilbert spaces, coupled with the exact inversion of the Abel integral operator, has been exploited in Refs. [17,22,31]. The importance of using orthogonal polynomials for the stable solution of problem (1) has been recognized for a long time [33]. The approximation of the unknown solution by Jacobi polynomials [6], Legendre polynomials [6,7] and Chebyshev polynomials [34,36] has been proposed for the inversion of the Abel integral operator.

In this paper, we present a new procedure for the computation of the inverse Abel transform. In Section 2 we prove that the Legendre coefficients of the solution $f(x)$ to problem (1) coincide with the Fourier coefficients of a suitable function associated with the data $g(x)$. The role of noise is studied in Section 3 where we focus on the regularization of problem (1) within the spectral cut-off scheme and introduce a suitably regularized solution $f_N^{(\varepsilon)}(x)$, ε being a parameter which represents the amount of noise on the data. Rigorous stability estimates for the proposed solution are then proved in the same Section 3, where we give upper bounds on the reconstruction error which depend on the smoothness properties of the solution and on the level of noise ε. The algorithm produced by this analysis results to be extremely simple and fast since the N coefficients of the regularized solution can be computed very efficiently by means of a single Fast Fourier Transform in $\mathcal{O}(N \log N)$ time. This attractive feature makes the algorithm particularly suitable for the Abel inversion of functions represented by samples, e.g., noisy experimental measurements, given on nonequispaced points of the Abel transform domain since the core of the computation can be simply performed by means of a nonuniform Fast Fourier Transform. Finally, in Section 4.1 we illustrate some numerical experiments which exemplify the theoretical analysis and give a taste of the stability of the algorithm for the computation of the inverse Abel transform for solutions with different smoothness properties and various levels of noise on the data.

2. Inversion of the Abel transform by Legendre expansion

For convenience, let us define the following intervals of the real line: $E \doteq (0, 1)$, $\Omega \doteq (-\pi, \pi)$. Consider the *shifted* Legendre polynomials $\overline{P}_n(x)$, which are defined by:

$$\overline{P}_n(x) = \sqrt{2n+1}\,P_n(2x-1), \tag{3}$$

where $P_n(x)$ denote the *ordinary* Legendre polynomials, defined by the generating function [20]:

$$\sum_{n=0}^{\infty} P_n(x)\,t^n = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}}. \tag{4}$$

The *shifted* Legendre polynomials $\{\overline{P}_n(x)\}_{n=0}^{\infty}$ form a complete orthonormal basis for $L^2(E)$. The (*shifted*) Legendre expansion of a function $f(x) \in L^2(E)$ reads:

$$f(x) = \sum_{n=0}^{\infty} c_n\,\overline{P}_n(x) \qquad (x \in E), \tag{5}$$

with coefficients $c_n = (f, \overline{P}_n)$ (where $(\cdot,\cdot)$ denotes the scalar product in $L^2(E)$):

$$c_n = \int_0^1 f(x)\,\overline{P}_n(x)\,\mathrm{d}x \qquad (n \geqslant 0). \tag{6}$$

We can now prove the following theorem which connects the Legendre coefficients of a function $f(y)$ with its Abel transform $g(x)$.

Theorem 1. *Let g denote the Abel transform (1) of the function $f \in L^2(E)$. Then the inverse Abel transform $f = (A^{-1}g)$ can be written as:*

$$f(x) = \sum_{n=0}^{\infty} c_n\,\overline{P}_n(x) \qquad (x \in E), \tag{7}$$

where $c_n = (-1)^n\sqrt{2n+1}\,\hat{\gamma}_n$ and the coefficients $\{\hat{\gamma}_n\}_{n=0}^{\infty}$ coincide with the Fourier coefficients (with $n \geq 0$) of the 2π-periodic auxiliary function $\eta(t)$ $(t \in \mathbb{R})$, whose restriction to the interval $t \in [-\pi, \pi)$ is given by

$$\eta(t) \doteq \frac{\operatorname{sgn}(t)}{2\pi\mathrm{i}}\,e^{\mathrm{i}t/2}\,g\!\left(\sin^2\frac{t}{2}\right) \qquad (t \in [-\pi, \pi)), \tag{8}$$