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Linearization criteria for systems of two second-order stochastic ordinary differential equations

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ABSTRACT

We provide the necessary and sufficient conditions for the linearization of systems of two second-order stochastic ordinary differential equations. The linearization criteria are given in terms of coefficients of the system followed by some illustrations. This paper gives a new treatment for the linearization of two second-order stochastic ordinary differential equations and with some examples.

1. Introduction

Stochastic ordinary differential equations (SODEs) include a stochastic component which describes the randomness within the differential equations. SODEs are in general nonlinear and their solutions are difficult to obtain. Various methods of solving differential equations involve applying a change of variables to transform a given differential equation in to another equation with known properties. The class of linear equations is known to be the simplest class of equations for which it is easier to find a solution, hence, the existence of the problem of transforming a given differential equation into a linear equation. This problem, called a linearization problem, is a particular case of an equivalence problem [1,3,7].

Linear SODEs play a role similar to that of linear equations in the deterministic theory of ordinary differential equations (ODEs). However, the change of variables in SODEs differs from that in ODEs due to the Itô formula. The transformation of nonlinear SODEs into linear ones via an invertible stochastic mapping prove to be useful in obtaining the closed form solutions [2,7,9,13]. In this paper, we present a general linearizability criteria for the systems of two second-order SODEs.

We consider the system of two second-order SODEs,

$$\begin{aligned} d\dot{X} &= f_1(t,X,Y,\dot{X},\dot{Y})\,dt + g_1(t,X,Y,\dot{X},\dot{Y})\,dW \\ d\dot{Y} &= f_2(t,X,Y,\dot{X},\dot{Y})\,dt + g_2(t,X,Y,\dot{X},\dot{Y})\,dW, \end{aligned} \tag{1}$$

where f_i and g_i, $(i = 1, 2)$ are deterministic functions and dW is the infinitesimal increment of the Wiener process [8]. System (1) is said to be linear if the functions f_i and g_i are linear functions with respect to variables X and Y and their respective derivatives. For the linearization problem one considers the class of equations equivalent to linear equations.

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Thus a linear system of two second-order SODEs has the form,

$$\begin{aligned} d\dot{X} &= \big(\alpha_{11}(t)X + \alpha_{12}(t)Y + \alpha_{13}(t)\dot{X} + \alpha_{14}(t)\dot{Y} + \alpha_{10}(t)\big)\,dt \\ &\quad + \big(\beta_{11}(t)X + \beta_{12}(t)Y + \beta_{13}(t)\dot{X} + \beta_{14}(t)\dot{Y} + \beta_{10}(t)\big)\,dW, \\ d\dot{Y} &= \big(\alpha_{21}(t)X + \alpha_{22}(t)Y + \alpha_{23}(t)\dot{X} + \alpha_{24}(t)\dot{Y} + \alpha_{20}(t)\big)\,dt \\ &\quad + \big(\beta_{21}(t)X + \beta_{22}(t)Y + \beta_{23}(t)\dot{X} + \beta_{24}(t)\dot{Y} + \beta_{20}(t)\big)\,dW. \end{aligned} \tag{2}$$

We can rewrite (2) in the form of first-order SODEs:

$$\begin{aligned} d\mathbf{X} &= \dot{\mathbf{X}}dt, \\ d\dot{\mathbf{X}} &= \big(A\mathbf{X} + B\dot{\mathbf{X}} + \mathbf{a}\big)\,dt + \big(F_1\mathbf{X} + F_2\dot{\mathbf{X}} + \mathbf{b}\big)\,dW, \end{aligned} \tag{3}$$

where $A(t)$, $B(t)$, F_i $(i = 1, 2)$ are 2×2 matrices; $\mathbf{a(t)}$, $\mathbf{b(t)}$ are vectors and

$$\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Similar to the treatment of ordinary differential equations (ODEs), the linearization problem involves finding a change of the dependent variables,

$$\bar{x} = \varphi(t, x, y),\ \bar{y} = \psi(t, x, y),\ \Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0,$$

which can transform the system of equations given in (1) into linear SODEs (2).

Lie [6] laid a foundation for the linearization criteria of the second-order ODEs via an invertible point transformation. He showed that the second-order ODE

$$\ddot{x} = f(t, x, \dot{x}), \tag{4}$$

is linearizable by a change of both the independent and dependent variables provided f is a polynomial of the third degree with respect to the first-order derivative,

$$\ddot{x} + F\dot{x}^3 + G\dot{x}^2 + H\dot{x} + L = 0,$$

where the coefficients $F(t, x)$, $G(t, x)$, $H(t, x)$ and $L(t, x)$ satisfy the conditions

$$K_1 = 3F_{tt} - 2G_{xt} + H_{xx} - 3F_tH + 3F_xL + 2G_tG - 3H_tF - H_xG + 6L_xF = 0 \tag{5}$$

$$K_2 = G_{tt} - 2H_{xt} + 3L_{xx} - 6F_tL + G_tH + 3G_xL - 2H_xH - 3L_tF + 3L_xG = 0.$$

Eq. (4) is also linearizable by a change of the dependent variable x provided $F = 0$ in which conditions (5) become

$$K_1 = (-2G_t + H_x)_x - G(-2G_t + H_x) = 0 \tag{6}$$

$$K_2 = (G_t - 2H_x)_t + H(G_t - 2H_x) + 3(L_x + GL)_x = 0.$$

Lie's linearizability criteria for second-order ODEs was extended to the system of second-order ODEs by the authors in [1,10,14] and the references therein. In Bagderina [1], a study of the linearization problem of the system of two second-order ODEs was completed.

Modifiying Lie's work for ODEs to SODEs has been done by [11], extended by [7] to the second-order SODEs and in [9,12,13], the conditions for the invertible transformations which linearize the jump-diffusion are obtained. The reducibility approach was used in [15] to study the linearization problem of stochastic differential equations (SDEs) with fractional Brownian motion. This work, however, misused the fractional Itô formula to derive the reducibility conditions of nonlinear fractional SDEs to linear fractional SDEs. This was reviewed and corrected in [5].

The rest of the paper is organized as follows: Section 2 discusses an equivalence transformation used to reduce the number of coefficients α_{ij} in system (2). In Section 3, the determining equations are derived in an Itô calculus context. These determining equations are non-stochastic. The linearization criteria for a system of two second-order SODEs are given in terms of coefficients of the system. The later part of the paper deals with the β_{ij}s also from Eq. (2) and the analysis of relations for them is given. The main result and Theorem are given in Section 3. Section 4 gives some examples and the conclusion is given in Section 5. To the best of our knowledge this is a new contribution on the linearization problem of systems of two second-order SODEs.

2. Equivalence transformation

We consider the transformation

$$\mathbf{X}_1 = C(t)\mathbf{X} + \mathbf{h}(t), \tag{7}$$

where $C = C(t)$ is a nonsingular matrix and $h(t)$ a vector.