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A remark on regularity criterion for the 3D Hall-MHD equations based on the vorticity

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ABSTRACT

In this paper we investigate the regularity criterion for the local-in-time classical solution to the three-dimensional (3D) incompressible Hall-magnetohydrodynamic equations (Hall-MHD). It is proved that the control of the vorticity alone can ensure the smoothness of the solution.

1. Introduction

In this paper, we are interested in the following 3D incompressible Hall-MHD equations in the whole space

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \mu \Lambda^{2\alpha} u + \nabla \pi = (B \cdot \nabla)B, \\ \partial_t B + (u \cdot \nabla)B + \nabla \times \big((\nabla \times B) \times B\big) + \nu \Lambda^{2\beta} B = (B \cdot \nabla)u, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(x,0) = u_0(x), \quad B(x,0) = B_0(x), \end{cases} \tag{1.1}$$

where $\mu \geq 0$, $\nu \geq 0$, $\alpha \geq 0$ and $\beta \geq 0$ are real parameters. Here $u = u(x,t) \in \mathbb{R}^3$, $B = B(x,t) \in \mathbb{R}^3$, $\pi = \pi(x,t) \in \mathbb{R}$ represent the unknown velocity field, the magnetic field and the pressure, respectively. The fractional Laplacian operator $\Lambda^{2\alpha}$ is defined through the Fourier transform, namely

$$\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

We remark that the fractional Laplacian models many physical phenomena such as overdriven detonations in gases [14]. It is also used in some mathematical models in hydrodynamics, molecular biology and finance mathematics, see for instance [16]. The classical Hall-MHD equations (namely (1.1) with $\alpha = \beta = 1$) were derived in [1] from either two-fluids or kinetic models which are useful in describing many physical phenomena in geophysics and astrophysics. The Hall-MHD is indeed needed for many problems such as magnetic reconnection in space plasmas [18], star formation [2], and also neutron stars [26]. When the Hall effect term $\nabla \times ((\nabla \times B) \times B)$ is neglected, system (1.1) reduces to the well-known magnetohydrodynamic (MHD) equations. The main difference between the magnetohydrodynamicequations (MHD) and Hall-MHD equations is the

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Hall term $\nabla \times ((\nabla \times B) \times B)$ included in $(1.1)_2$ due to Ohm's law, which plays an important role in magnetic reconnection which is happening in the case of large magnetic shear.

We remark that there is a growing literature devoted to studying the regularity criteria for the fluid mechanics, such as the Navier–Stokes equations, MHD equations, micropolar equations and so on. As a result, it is almost impossible to be exhaustive in this introduction, and we just name a few some notable works [3,4,6–8,13,15,19,20,22,28,31]. Due to the physical applications and mathematical significance, there is a considerable body of literature on the classical Hall-MHD equations and the fractional Hall-MHD equations. The classical Hall-MHD equations have been studied systematically by Lighthill [24] followed by Campos [5]. Later, the global existence of weak solutions and the local well-posedness of classical solution to the classical Hall-MHD were established by Chae–Degond–Liu in [9]. The Prodi–Serrin conditions type criterion and small data global existence results for the solutions to the system (1.1) were established by Chae and Lee in the recent work [10], which was improved by Fan et al. [17] and Wan and Zhou [27]. The local well-posedness for the system (1.1) with only $\Lambda^{2\beta}B$ ($\beta > 1/2$ and $\mu = 0$) was obtained in [12] via the Besov space technique, while the case $\beta \le 1/2$ is still unknown. According to the minimal requirements for the fractional Navier–Stokes equations and the simple Hall problem, the system (1.1) indeed admits a unique global classical solution as long as $\alpha \ge \frac{5}{4}$ and $\beta \ge \frac{7}{4}$ (see [29]). However, whether the local smooth solution of the system (1.1) with $\alpha < \frac{5}{4}$ and $\beta < \frac{7}{4}$ can exist globally is a challenging open problem in the mathematical fluid mechanics. In the absence of a global well-posedness theory, the development of regularity criteria is of major importance for both theoretical and practical purposes. For this reason, there have been many studies about the regularity criteria to the corresponding system (see [9–11,29,30] and reference therein). It should be noticed that, for the MHD equations, the velocity field plays a more dominant role than the magnetic field does. Therefore, regularity criteria based only on the velocity were established (see for instance [20,31]).

Now a nature question is what may happen about the Hall-MHD equations. Actually, due to the presence of the Hall term $\nabla \times ((\nabla \times B) \times B)$ in the Hall-MHD equations, it seems very difficult to establish regularity criteria based only on the velocity or the vorticity. The goal of this paper is to establish a regularity criterion of the solution to the Hall-MHD equations to remain smooth for all time based only on the vorticity. In other word, our results demonstrate that the magnetic field plays less dominant role than the velocity field does in the regularity theory of solutions to the Hall-MHD equations. More precisely, the main result reads as follows.

Theorem 1.1. *Consider the system (1.1) with $1 < \alpha = \beta < \frac{7}{4}$. Assume that $(u_0, B_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ ($s > \frac{7}{2}$) with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. Let (u, B) be a local corresponding smooth solution of the system (1.1). If one of the following three conditions holds*

$$\int_0^T \|\omega(t)\|_{L^\infty}^{\frac{\alpha}{\alpha-1}}\,dt < \infty; \tag{1.2}$$

$$\int_0^T \|\omega(t)\|_{\dot{B}^0_{\infty,\alpha}}^{\frac{\alpha}{\alpha-1}}\,dt < \infty; \tag{1.3}$$

$$\int_0^T \|\omega(t)\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{2\alpha}{2\alpha-2-\delta}}\,dt < \infty, \quad 0 < \delta < \min\{2\alpha-2,\,1\}, \tag{1.4}$$

then the solution (u, B) can be extended past time T, where ω is the vorticity of the velocity u, namely, $\omega = \nabla \times u$.

Remark 1.2. The proof of Theorem 1.1 depends on a new quantity introduced by the recent work [21]. The authors [21] established the regularity criterion (1.4), whose proof relies heavily on the Besov space techniques. However, we chose not to apply the Littlewood–Paley decomposition to the system itself but to use some simple interpolation inequalities involving the Besov norms, which leads to the simplicity and readability of our paper.

Remark 1.3. As the global well-posedness can be established for the system (1.1) with $\alpha \ge \frac{5}{4}$ and $\beta \ge \frac{7}{4}$, it suffices to consider the case $0 < \alpha = \beta < \frac{7}{4}$.

2. The proof of Theorem 1.1

This section is devoted to the proof of the Theorem 1.1. Before proving the theorem, we first introduce the following conventions and notations which will be used. Throughout the paper, C stands for some real positive constants which may be different in each occurrence. We shall sometimes use the natation $A \lesssim B$ which stands for $A \le CB$. Without loss of generality, we set $\mu = \nu = 1$ in the rest of the paper since their sizes do not play any role in our analysis. The existence and uniqueness of local smooth solutions can be done as in the case of the Euler and Navier–Stokes equations, thus may assume that (u, B) is smooth enough in the interval $[0, T)$. We will establish a priori bounds that will allow us to extend (u, B) past time T under (1.2), (1.3) or (1.4).

Taking the inner product of $(1.1)_1$ with u and $(1.1)_2$ with B, one may get by using several cancellation properties

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + 2\int_0^t (\|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\alpha B\|_{L^2}^2)(\tau)\,d\tau = \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2. \tag{2.1}$$