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Multiphysics discontinuous Galerkin method for a poroelasticity model☆

Zhihao Ge[a,b,*], Mengxia Ma[a]

[a] School of Mathematics and Statistics, Henan University, Kaifeng 475004, PR China
[b] Institute of Applied Mathematics, Henan University, Kaifeng 475004, PR China

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ABSTRACT

In this paper, we develop and analyze a multiphysics discontinuous Galerkin method for a poroelasticity model, which describes the dynamics of poro-elastic materials under an applied mechanical force on the boundary. And we prove that the multiphysics discontinuous Galerkin method is absolutely stable for all positive mesh size h. Also, we propose a time-stepping algorithm which decouples the reformulated poroelasticity model at each time step into two sub-problems, one of which is a generalized Stokes problem for the displacement vector field along with a pseudo-pressure and the other is a diffusion problem for the pseudo-pressure field. And we give the optimal order error estimates in the energy norm. Finally, we give the numerical examples to verify the theoretical results.

1. Introduction

In recent years, several variations of finite element methods for poroelasticity model have been proposed which exhibit special convergence, conservation, and approximation properties (cf. [2,3]). A poroelasticity material is a fluid-solid interaction system at pore scale and poromechanic is a branch of continuum mechanics and acoustics that studies the behavior of fluid-saturated porous materials. If the solid is an elastic material, then the subject of the study is known as poroelasticity. Poroelastic materials include soil, polymer gels, and medicine pills, just name a few, which not only exhibit an important state of matter found in a wide variety of mechanical, biomedical and chemical systems(cf. [5,6,14,15] and the references therein), but also possess some fascinating properties, in particular, they display thixotropy which means that they become fluid when agitated, but resolidify when resting. In [9,10,12] the authors proposed and analyzed a semi-discrete and a fully discrete mixed finite element method which simultaneously approximate the pressure and its gradient along with the displacement vector field. However, when the constrained specific storage coefficient $\kappa_0 = 0$, the CG/mixed method may exhibit a "locking phenomenon" in poroelasticity as explained in the [11]. To overcome the above difficulty and reveal the multi-physical process of the original poroelasticity model, we develop a multiphysics discontinuous Galerkin method for the poroelasticity model, which is a novel method and very different from the DG/mixed scheme of [11]. Discontinuous Galerkin method has the ability to deal with the discontinuity flexibly, and the finite element spaces do not need satisfy the so-called inf–sup condition. Also, discontinuous Galerkin method is easy to be adaptive and parrel. Due to the above obvious advantages, the discontinuous Galerkin method becomes an active research area in recent years (cf. [1,4,8,13]).

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* Corresponding author at: School of Mathematics and Statistics, Henan University, Kaifeng 475004, PR China.
E-mail addresses: 13663786282@163.com, zhihaoge@henu.edu.cn (Z. Ge), meng601203@163.com (M. Ma).

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In the work, we aim to design the multiphysics discontinuous Galerkin (DG) method, the key idea is to introduce some interior penalty terms in mixed forms of the proposed DG method and the main difficulty is to prove the discrete inf–sup condition. To the end, we reformulate the poroelasticity model and decoupled (or coupled) the reformulated problem at each time step into two sub-problems – a generalized Stokes problem for the displacement vector field along with a pseudo-pressure and a diffusion problem for other pseudo-pressure field.

The remainder of this paper is organized as follows. In Section 2, we present a complete PDE model and its multiphysics reformulation, and introduce some new variables to change the forms of problems. In Section 3, we define the bilinear forms and propose the multiphysics discontinuous Galerkin method. In Section 4, we prove the coerciveness of the discrete bilinear form. Then, in Section 5, we prove that the optimal order error estimates in the energy norm are established. Finally, we give some numerical tests to show the effective of the proposed approach and method, and there is no "locking phenomenon" in our numerical method.

2. PDE model and its multiphysics reformulation

The quasi-static poroelasticity model to be studied in this paper is given by

$$-\text{div}\ \sigma(\mathbf{u}) + \alpha\nabla p = \mathbf{f} \quad \text{in } \Omega_T := \Omega \times (0,T) \subset \mathbb{R}^d \times (0,T), \tag{2.1}$$

$$(\kappa_0 p + \alpha \text{div}(\mathbf{u}))_t + \text{div}\mathbf{v}_f = \phi \quad \text{in } \Omega_T, \tag{2.2}$$

where

$$\sigma(\mathbf{u}) := \mu\varepsilon(\mathbf{u}) + \lambda\text{div}\mathbf{u}I, \varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \tag{2.3}$$

$$\mathbf{v}_f := -\frac{K}{\mu_f}(\nabla p - \rho_f \mathbf{g}), \tag{2.4}$$

and $\mathbf{u}$ denotes the displacement vector of the solid and p denotes the pressure of the solvent, $\mathbf{f}$ is the body force, I denotes the $d \times d$ identity matrix and $\varepsilon(\mathbf{u})$ is the strain tensor. The parameters are Lamé constants λ, μ, the permeability tensor K, the solvent viscosity μ_f, Biot–Willis constant α, and the constrained specific storage coefficient κ_0. In addition, $\sigma(\mathbf{u})$ is called the (effective) stress tensor, $\widehat{\sigma}(\mathbf{u}, p) := \sigma(\mathbf{u}) - \alpha pI$ is the total stress tensor. $\mathbf{v}_f$ is the volumetric solvent flux and (2.4) is the well-known Darcy's law. We assume that $\rho_f \not\equiv 0$, which is a realistic assumption.

To close the above system, suitable boundary and initial conditions must be prescribed. The following set of boundary and initial conditions are considered in this paper:

$$\widehat{\sigma}(\mathbf{u}, p)\mathbf{n} = \sigma(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n} = \mathbf{f}_1 \quad \text{on } \partial\Omega_T := \partial\Omega \times (0,T), \tag{2.5}$$

$$\mathbf{v}_f \cdot \mathbf{n} = -\frac{K}{\mu_f}(\nabla p - \rho_f \mathbf{g}) \cdot \mathbf{n} = -\phi_1 \quad \text{on } \partial\Omega_T, \tag{2.6}$$

$$\mathbf{u} = \mathbf{u_0}, p = p_0 \quad \text{on } \Omega \times \{t = 0\}. \tag{2.7}$$

Now, we introduce new variables

$$q := \text{div}\mathbf{u}, \quad \eta := \kappa_0 p + \alpha q, \quad \xi := \alpha p - \lambda q.$$

It is easy to check that

$$p = k_1\xi + k_2\eta, \quad q = k_1\eta - k_3\xi, \tag{2.8}$$

where

$$k_1 := \frac{\alpha}{\alpha^2 + \lambda\kappa_0}, \quad k_2 := \frac{\lambda}{\alpha^2 + \lambda\kappa_0}, \quad k_3 := \frac{\kappa_0}{\alpha^2 + \lambda\kappa_0}. \tag{2.9}$$

Then the system (2.1)–(2.7) can be written as

$$-\mu\text{div}\varepsilon(\mathbf{u}) + \nabla\xi = \mathbf{f} \quad \text{in } \Omega_T, \tag{2.10}$$

$$k_3\xi + \text{div}\mathbf{u} = k_1\eta \quad \text{in } \Omega_T, \tag{2.11}$$

$$\eta_t - \frac{1}{\mu_f}\text{div}[K(\nabla(k_1\xi + k_2\eta) - \rho_f\mathbf{g})] = \phi \quad \text{in } \Omega_T, \tag{2.12}$$

$$\sigma(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n} = \mathbf{f}_1 \quad \text{on } \partial\Omega_T, \tag{2.13}$$

$$-\frac{K}{\mu_f}(\nabla p - \rho_f\mathbf{g}) \cdot \mathbf{n} = -\phi_1 \quad \text{on } \partial\Omega_T, \tag{2.14}$$