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Complexity and algorithms for the connected vertex cover problem in 4-regular graphs☆

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ABSTRACT

In the *connected vertex cover (CVC) problem*, we are given a connected graph G and required to find a vertex cover set C with minimum cardinality such that the induced subgraph $G[C]$ is connected. In this paper, we restrict our attention to the *CVC* problem in 4-regular graphs. We proved that the *CVC* problem is still NP-hard for 4-regular graphs and gave a lower bound for the problem. Moreover, we proposed two approximation algorithms for *CVC* problem with approximation ratio $\frac{3}{2}$ and $\frac{4}{3}+O(\frac{1}{n})$, respectively.

1. Introduction

All the graphs considered in this paper are simple graphs without loops and multi-edges. Let G be a graph, $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. For $v \in V(G)$, *the degree* $d(v)$ is equal to the number of neighbors of v. Denote by $\Delta(G)$ the maximum degree of G. If all the vertices of G have the same degree k, then G is called k-regular. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. A complete graph with n vertices is denoted by K_n. A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph. A complete bipartite graph is a bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. For a subset $X \subseteq V(G)$, the subgraph induced by X is denoted by $G[X]$. The set of edges between X and Y is denoted by $E(X, Y)$ for $X, Y \subseteq V(G)$.

The vertex cover (VC) problem is a classic problem in combinatorial optimization and operations research. Given a graph G, the goal of *VC* is to find a subset of vertices $C \subset V$ with minimum cardinality such that every edge in the graph is incident to a vertex in C. *The connected vertex cover (CVC) problem* is a variation of the vertex cover problem, which aims at finding a vertex cover C with minimum cardinality such that $G[C]$ is connected.

The *CVC* problem was first introduced by Garey and Johnson [6] in 1977, which finds many applications in real life. For example, in the field of wireless network design [10], the vertices and the edges represent the network nodes and transmission links, respectively. Some relay stations will be placed on some network nodes such that they form a connected subnetwork and every transmission link is incident to a relay station. We want to minimize the number of relay stations. This is exactly the connected vertex cover problem.

The *CVC* problem is NP-hard and a 2-approximation algorithm is widely known [1,12], but it is NP-hard to be approximated within ratio $10\sqrt{5}-21$ [3]. In recent years, a great deal of efforts have been devoted to this problem on various

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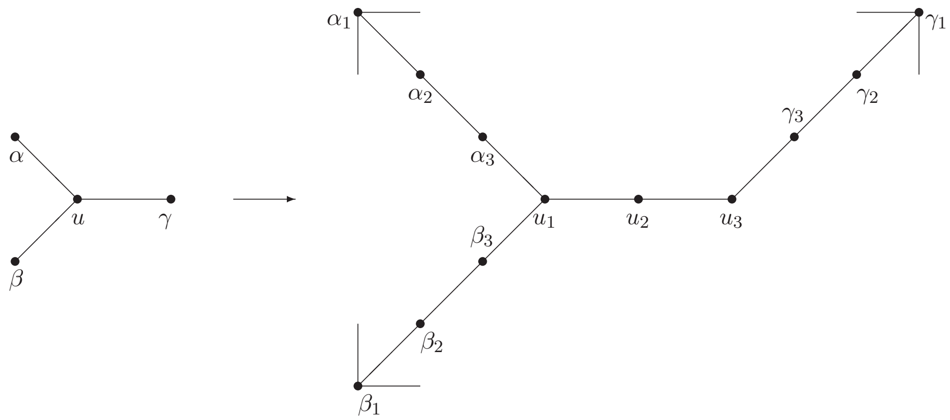

Fig. 1. The transformation of the vertices in the reduction from a cubic graph H to a (2, 3)-regular graph G.

classes of graphs. In [6], Garey and Johnson proved that this problem is NP-complete in planar graphs of maximum degree 4. Then researchers proved that it is also NP-complete in planar bipartite graphs of maximum degree 4 [3], in planar bi-connected graphs of maximum degree 4 [11] and in 3-connected graphs [14]. Ueno et al. [13] proved, however, that this problem can be solved in polynomial time for graphs with no vertex degree exceeding 3. Also, Zhang et al. [15] gave the first polynomial time approximation scheme for the connected vertex cover problem in unit disk graphs.

Recently the fixed-parameter tractability of the connected vertex cover problem with respect to the vertex cover size or to the treewidth of the input graph has been widely studied; see e.g., [3,7–10]. In [2], the authors showed that the connected vertex cover problem is polynomial-time solvable in chordal graphs and proved that the problem is APX-complete in bipartite graphs of maximum degree 4, even if each vertex of one block of the bipartition has a degree at most 3. On the other hand, if each vertex of one block of the bipartition has a degree at most 2 (and the vertices of the other part have an arbitrary degree), then the problem is polynomial time solvable. They also showed that the connected vertex cover problem is $\frac{5}{3}$-approximable in any class of graphs where the vertex cover problem is polynomial time solvable (in particular in bipartite graphs, or more generally in perfect graphs). Then, they presented a polynomial approximation scheme for the connected vertex cover problem in planar graphs.

In this paper, we shall investigate the *CVC* problem for 4-regular graphs from an algorithmic point of view. We proved that the *CVC* problem is NP-hard for 4-regular graphs and gave a lower bound for the minimum size of a *CVC*. Then we gave a 1.5-approximation algorithm by slightly modifying a previously known algorithm, and at last we propose a better approximation algorithm with approximation ratio $\frac{4}{3}+O(\frac{1}{n})$ for the *CVC* problem in 4-regular graphs.

In Section 2, we show that *CVC* is NP-hard for 4-regular graphs. In Section 3, we present a lower bound for *CVC* problem and propose two approximation algorithms. Conclusions and future works are given in Section 4.

2. The *CVC* problem is NP-hard for 4-regular graphs

In this section, we show the NP-hardness of *CVC* problem on 4-regular graphs, by reducing another NP-hard problem - *VC* problem on cubic graphs, to it. First, we need the following definition.

Definition 1. For a graph G, if the degree of any vertex of G is k or l, then the graph G is called a (k, l)-regular graph.

Theorem 1. *The CVC problem is NP-hard for 4-regular graphs G.*

Proof. We begin the proof with the following two claims.

Claim 1 The *VC* problem is NP-hard for (2, 3)-regular graphs.

We reduce the *VC* problem for cubic graphs, which is known to be NP-hard [5], to the *VC* problem for (2, 3)-regular graphs.

Let $H=(V,E)$ be a cubic graph. We construct a graph $G=(V',E')$ as follows. For each vertex $u\in V$, we split it into three vertices u_1, u_2 and u_3 such that two neighbors of u are connected to u_1 and the third one is connected to u_3. Add two edges (u_1, u_2) and (u_2, u_3). G is a (2, 3)-regular graph, see Fig. 1.

Let $S\subseteq V$ be a vertex cover set of H. We construct a corresponding vertex cover set S' in G. If $u\in S$, then $u_1, u_3\in S'$. If $u\notin S$, then $u_2\in S'$. Clearly that S' is a vertex cover set of G and $|S'|=|S|+|V|$.

On the other hand, let S' be a vertex cover set in G. We construct another vertex cover set S'' in G such that $|S''|\le|S'|$ and $|S''\cap\{u_1, u_2, u_3|: u\in V\}|\le 2$. Initially, let $S''=S'$. It is easy to see that at least one vertex in $\{u_1, u_2, u_3\}$ must be in S'. If the three vertices are all in S', $S''=S''\backslash\{u_2|u_1,u_2,u_3\in S'\}$. Clearly, S'' is also a vertex cover set in G. Now, we construct a corresponding set S in H as follows. If $|S''\cap\{u_1,u_2,u_3|:u\in V\}|=1$, then $u\notin S$. If $|S''\cap\{u_1,u_2,u_3|:u\in V\}|=2$, then $u\in S$. In the first case, exactly one vertex of $\{u_1, u_2, u_3\}$ is in S'', i.e. u_2, then the vertices which are the neighbors of u_1 and u_3