Applied Mathematics and Computation 301 (2017) 140–154

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Computing two dimensional Poincaré maps for hyperchaotic dynamics

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ARTICLE INFO

Keywords:
Poincaré map
Hyperchaos
Recurrence Quantification
LE-I & LE-II
Mean recurrence time statistics

ABSTRACT

Poincaré map (PM) is one of the felicitous discrete approximation of the continuous dynamics. To compute PM, the discrete relation(s) between the successive point of interactions of the trajectories on the suitable Poincaré section (PS) are found out. These discrete relations act as an amanuensis of the nature of the continuous dynamics. In this article, we propose a computational scheme to find a hyperchaotic PM (HPM) from an equivalent three dimensional (3D) subsystem of a 4D (or higher) hyperchaotic model. For the experimental purpose, a standard four dimensional (4D) hyperchaotic Lorenz-Stenflo system (HLSS) and a five dimensional (5D) hyperchaotic laser model (HLM) is considered. Equivalent 3D subsystem is obtained by comparing the movements of the trajectories of the original hyperchaotic systems with all of their 3D subsystems. The quantitative measurement of this comparison is made promising by recurrence quantification analysis (RQA). Various two dimensional (2D) Poincaré mas are computed for several suitable Poincaré sections for both the systems. But, only some of them are hyperchaotic in nature. The hyperchaotic behavior is verified by positive values of both one dimensional (1D) Lyapunov Exponent (LE-I) and 2D Lyapunov Exponent (LE-II). At the end, similarity of the dynamics between the hyperchaotic systems and their 2D hyperchaotic Poincaré maps (HPM) has been established through mean recurrence time (MRT) statistics for both of 4D HLSS and 5D HLM and the best approximated discrete dynamics for both the hyperchaotic systems are found out.

1. Introduction

Most of the complex dynamics of continuous phenomena can be properly manifested in high dimensional attractor [1–9]. Hyperchaos is one of such dynamics which requires the embedding dimension at least four [10–19]. From the dynamical

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http://dx.doi.org/10.1016/j.amc.2016.12.026

aspect, a system with at least two positive Lyapunov exponents (LE) [20–22], one null exponent and one negative exponent assures hyperchaos. Continuous as well as discrete hyperchaotic phenomena can be described by a set of differential and difference equations, respectively. In both the cases, long term dynamics can be characterized by some known standard qualitative analysis. However, discrete dynamics is more easy to quantify the dynamical behaviour compare to continuous one. So, several attempts have been made to approximate the continuous dynamics by a discrete dynamics. PM [23–27] is one of the best approximation method in this concern.

In the PM approximation method, a plane surface (known as PS) is projected orthogonally on the flow of the trajectories of the attractors. Then, an N dimensional trajectory $X = (x)^T \in R^N$ of the system $\frac{dX}{dt} = f(X, P)$ ($P = (p)^T \in R^m$ being the parameter) intersects the PS at some $N-1$ dimensional points. By arranging the points based on their successive intersection with the PS, we can get a sequence of $N-1$ dimensional points $\{U_n = (u_n)^{N-1} \in R^{N-1}\}$, where $(u_n) = (u_n^1, u_n^2, \ldots, u_n^{N-1})$. Thus, $N-1$ dimensional PM described by the equations $u_{n+1} = S_k(u_n)$ $(k = 1, 2, 3, \ldots)$ are obtained. These actually represent the hyper surfaces in $N-1$ dimension. In order to establish the existence of the PM, we need to verify whether the intersection of the surfaces exists. Indeed no visual analysis is there to verify the intersection of the high dimensional (> 4) surfaces, the only way to detect the point of intersection is to solve the nonlinear equations $u_{n+1} = S_k(u_n)$. However, the process of solving $N-1$ nonlinear equations involves a large computational procedure. Moreover, the equations can possess non-feasible solutions in the sense that either these equations are of complex nature or the roots of the equations are complex number. So, computation of PM is not reasonable to approximate the high dimensional continuous dynamics. However, if nature of flow of the trajectories in higher dimensional dynamics and any of its 3D subsystem are found to be similar, the trajectories of both the systems make almost same kind of interactions on the plane of section. In fact, almost similar approximation can be made through a 2D PM of the corresponding 3D subsystem. In this concern, recurrence plot (RP) [28–32] is one of the effective method which can describe the behavior of the trajectories, whatever the dimension of the attractor.

In RP method, all the points of attractors are mapped into a matrix whose elements are either ‘1’ or ‘0’. A function R_{ij} from the attractor $P = \{\vec{x_i}\}_{i=1}^{N} \subset \mathbb{R}^n$ to the set {0,1} is defined as

$$R_{ij} = \begin{cases} 1, & \text{if } \|\vec{x_i} - \vec{x_j}\| < \epsilon \\ 0, & \text{otherwise.} \end{cases}$$

The radius of the neighborhood is denoted by ϵ. In practice, ϵ is taken as 10% of the diameter of the attractor [30]. The matrix $(R_{ij})_{N \times N}$ is called recurrence matrix. The diagram constructed by plotting the matrix $(R_{ij})_{N \times N}$ with two colors which correspond 1 and 0, respectively. In an RP, two points fall in ϵ-neighborhood correspond a colored point, otherwise it shows white. Two major structures found in RP diagonal line and vertical/horizontal line [28–32] Diagonal line explains parallel motion of the trajectories. Horizontal/vertical lines describe how trajectories are trapped in a position [30]. Some perpendicular to the diagonals and bowed line structure can also be observed in RP [30], which are in fact not concerned in this context. Thus, by constructing the RP of the higher order dynamical system and its 3D subsystems, it is easy to check whether their trajectories behave alike or not. In fact, it can describes similarity and dissimilarity between the trajectories of the original dynamical system and its subsystems, respectively. On the other hand, recurrence quantification analysis (RQA) has been introduced to quantify the dynamical behavior from the structure of RP. So, RQA can be applied to compare the trajectory's movements between two attractors. Two RQA measures - determinism (DET) and laminarity (LAM) [30] are very much effective in this purpose.

After recognizing the equivalent 3D subsystem, the fitted surfaces to the points on the PS take the form: $u_{n+1}^k = s_k((u_n^k)), k = 1, 2$. Since the surfaces lie in 3D, their intersection (if exists) can be easily visualized in 3D. This proves the existence of 2D PM. Now this 2D map may or may not possesses hyperchaos [33]. So, study of 1D Lyapunov exponent (LE - I) and 2D Lyapunov exponent (LE - II) [33] is very much essential. Theory of variational equation and exterior product or wedge product [33] are used to calculate LE - I and LE - II. Positive LE - I describes only chaotic behavior of the 2D map [33]. So, LE - II is calculated and positivity of LE - II along with the positive LE - I confirms the existence of hyperchaos. Since 1D PM no longer reflects the hyperchaotic dynamics [33], further reduction of 2D PM is not required at all.

Now for each possible PS, existence of many hyperchaotic 2D Poincaré maps is obvious. From the collection of all such HPM's, it is necessary to investigate dynamical similarity between the hyperchaotic maps and the corresponding high dimensional system. Mean recurrence time (MRT) statistics [34–36] can quantified such similarity. MRT statistic is such a measure, which reveals average features of the non-recurrent points of the attractor. Recurrent time [34] means a duration which is taken by a trajectory to come closer again. So, study of the distribution of MRT statistics [34–36] with respect to different segments of the trajectories gives a detail features about the dynamics of the attractor. In fact, by studying the nature of the distribution of MRT of two systems (continuous/discrete), similarity of the dynamics can be established.

This article is organized as follows: In Section 2, we have described a new computational scheme to compute a 2D Poincaré map for a high (≥ 4) dimensional hyperchaotic system. Section 3 describes the method of recognizing a 3D susbystem which has similar trajectory movements as that of the 4D HLSS. In Section 4, 2D hyperchaotic Poincaré maps are found out with respect to suitable Poincaré sections. The suitable Poincaré section are chosen by observing the nature of the 3D subsystem given by Section 3. The hyperchaotic nature is investigated by LE-I and LE-II. The general equation of such 2d hyperchaotic maps are also investigated. Section 5 discusses about the equivalence dynamics between the 2D HPM and the 4D HLSS. We have further computed a proper 2D HPM for a 5D HLM. The numerical results are briefly discussed in the Section 6. Section 7 is the conclusion.