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Blow-up results and soliton solutions for a generalized variable coefficient nonlinear Schrödinger equation

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ABSTRACT

In this paper, by means of similarity transformations we study exact analytical solutions for a generalized nonlinear Schrödinger equation with variable coefficients. This equation appears in literature describing the evolution of coherent light in a nonlinear Kerr medium, Bose–Einstein condensates phenomena and high intensity pulse propagation in optical fibers. By restricting the coefficients to satisfy Ermakov–Riccati systems with multiparameter solutions, we present conditions for existence of explicit solutions with singularities and a family of oscillating periodic soliton-type solutions. Also, we show the existence of bright-, dark- and Peregrine-type soliton solutions, and by means of a computer algebra system we exemplify the nontrivial dynamics of the solitary wave center of these solutions produced by our multiparameter approach.

1. Introduction

The study of the nonlinear Schrödinger equation (NLS) with real potential V

$$i\psi_t = -\frac{1}{2}\Delta\psi + V(\boldsymbol{x},t)\psi + \lambda(\boldsymbol{x},t)|\psi|^{2s}\psi, \qquad \psi(0,\boldsymbol{x}) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x}\in\mathbb{R}^n, \qquad \Delta = \sum_{j=1}^{n}\partial_{x_j x_j} \tag{1.1}$$

has been studied extensively not only for its role in physics, such as in Bose–Einstein condensates and nonlinear optics, but also for its mathematical complexity (for a review of the several results available see [1,3,4,12,13,18,19,27,32,33,47,51,52,58,63,68]). For the case $\lambda = -1$, $V \equiv 0$, and $ns < 2$ (subcritical case) Weinstein [65] proved that if $\varphi \in H^1$, then ψ exists globally in H^1. It is also known (see [13,21,58] for a complete review) that NLS for critical ($ns = 2$) and supercritical ($ns > 2$) cases present solutions that become singular in a finite time in L^p for some finite p. In [22] singular solutions of the subcritical NLS were presented in L^p.

In [10] it was proved that if $\varphi \in \Sigma = \{f \in H^1(\mathbb{R}^n) : \boldsymbol{x} \to |\boldsymbol{x}| f(\boldsymbol{x}) \in L^2(\mathbb{R}^n)\}$, $V(\boldsymbol{x},t)$ is real, locally bounded in time and subquadratic in space, and $\lambda \in \mathbb{R}$, then the solution of the Cauchy initial value problem exists globally in Σ, provided that $s < 2/n$ or $s \geq 2/n$ and $\lambda \geq 0$. Also, in [10] it was shown that if $V(\boldsymbol{x},t) = b(t)x_j^2$, $b(t) \in C(\mathbb{R};\mathbb{R})$ in (1.1), then there exist blow-up solutions if $\lambda < 0$ and $s = 2/n$. The proof uses the generalized Melher's formula introduced in [15]. In [44] and [59] a generalized pseudoconformal transformation (lens transform in optics [60]) was presented. In this paper, as a first

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main result we will use a generalized lens transformation to construct solutions with finite-time blow-up in L^p norm for $1 \leq p \leq \infty$ of the general variable coefficient nonlinear Schrödinger:

$$i\psi_t = -a(t)\psi_{xx} + (b(t)x^2 - f(t)x + G(t))\psi - ic(t)x\psi_x - id(t)\psi + ig(t)\psi_x + h(t)|\psi|^{2s}\psi. \quad (1.2)$$

In modern nonlinear sciences some of the most important models are the variable coefficient nonlinear Schrödinger-type ones. Applications include long distance optical communications, optical fibers and plasma physics, see [4,5,7–9,11,14,17,24–26,29,39,43,45,46,49,54,55,62,64,67] and references therein.

If we make $a(t) = \Lambda/4\pi n_0$, Λ being the wavelength of the optical source generating the beam, and choose $c(t) = g(t) = 0$, then (1.2) models a beam propagation inside of a planar graded-index nonlinear waveguide amplifier with quadratic refractive index represented by $b(t)x^2 - f(t)x + G(t)$, and $h(t)$ represents a Kerr-type nonlinearity of the waveguide amplifier, while $d(t)$ represents the gain coefficient. If $b(t) > 0$ [45] (resp. $b(t) < 0$, see [49]) in the low-intensity limit, the graded-index waveguide acts as a linear defocusing (focusing) lens.

Depending on the selections of the coefficients in Eq. (1.2), the applications vary in very specific problems (see [62] and references therein):

- Bose–Einstein condensates [27]: $b(\cdot) \neq 0$, a, h constants and other coefficients are zero.
- Dispersion-managed optical fibers and soliton lasers [29,54,55]: $a(\cdot)$, $h(\cdot)$, $d(\cdot) \neq 0$ are respectively dispersion, nonlinearity and amplification, and the other coefficients are zero. $a(\cdot)$ and $h(\cdot)$ can be periodic as well, see [2,40].
- Pulse dynamics in the dispersion-managed fibers [39]: $h(\cdot) \neq 0$, a is a constant and other coefficients are zero.

In this paper to obtain the main results we use a fundamental approach consisting of the use of similarity transformations and the solutions of Riccati Ermakov systems with several parameters inspired by the work in [38]. Similarity transformations have been a very popular strategy in nonlinear optics since the lens transform presented by Talanov [60]; extensions and applications of this approach have been presented by Rypdal and Rasmussen [50–53], see also more recent contributions by [44,59]. Also for related work on transformations using Lie groups and Lie algebras we refer the reader to [6,23,42,48,66]. Applications include nonlinear optics, Bose–Einstein condensates, integrability of NLS and quantum mechanics, see for example [5,6,10,34] and references therein. Marhic in 1978 introduced (probably for the first time) a one-parameter $\{\alpha(0)\}$ family of solutions for the linear Schrödinger equation of the one-dimensional harmonic oscillator; the use of an explicit formulation (classical Melher's formula [20,41]) for the propagator was fundamental. The solutions presented by Marhic constituted a generalization of the original Schrödinger wave packet with oscillating width. Also, in [15] a generalized Melher's formula for a general linear Schrödinger equation of the one-dimensional generalized harmonic oscillator of the form (1.2) with $h(t) = 0$ was presented. For the latter case in [31,35], multiparameter solutions in the spirit of Marhic in [38] have been presented. The parameters for the Riccati system arose originally in the process of proving convergence to the initial data for the Cauchy initial value problem (1.2) with $h(t) = 0$ and in the process of finding a general solution of a Riccati system [56]. Ermakov systems with solutions containing parameters [31] have been used successfully to construct solutions for the generalized harmonic oscillator with a hidden symmetry [35], and they have also been used to present Galilei transformation, pseudoconformal transformation and others in a unified manner, see [35]. More recently they have been used in [36] to show spiral and breathing solutions and solutions with bending for the paraxial wave equation. In this paper, as a second main result we introduce a family of Schrödinger equations presenting periodic soliton solutions by using multiparameter solutions for Riccati–Ermakov systems. Further, as a third main result we show that these parameters provide a control on the dynamics of solutions for equations of the form (1.2). These results should deserve numerical and experimental studies.

This paper is organized as follows: In Section 2, as an application of a generalized lens transformation and multiparameter solutions for Riccati systems we present conditions to obtain solutions with singularity in finite time in L^p norm, $1 \leq p \leq \infty$ for (1.2). Also, we show that through this more general parameter approach we can obtain the same L^∞ solutions with finite-time blow-up for standard NLS presented in [15] and finite-time blow-up for NLS with quadratic potential. In Section 3, we present a family of soliton solutions for (1.2) presenting bright- and dark-type solitons; this family includes the standard NLS models. This family has multiparameter solutions coming from solutions of a related Ermakov system, extending the results presented in [57], where a Riccati system was used. By the use of these parameters the dynamics of periodic solutions for (1.2) show bending properties, see Figs. 1 and 2. In Section 4, again, as an application of generalized lens transformations and an alternative approach to solve the Riccati system (A.1)–(A.6) we present how the parameters provide us with a control on the center axis of the solution of bright and dark soliton solutions for special coefficients in (1.2). Figs. 3 and 4 show the bending propagation of the solutions after introducing parameters, extending the results presented in [36,57] to (1.2). Also we show that it is possible to construct a transformation that reduces (1.2), with $a(t) = l_0 = \pm 1$ and $G(t) = 0$, to standard NLS with convenient initial data (Lemma 4) in order to assure existence and uniqueness of classical solutions (Proposition 1). As an application we show how the dynamics of the Peregrine soliton solutions of the nonlinear Schrödinger equation consider change when the dissipation, $d(t)$, and the nonlinear term, $h(t)$ change, see Figs. 5–8. We have also prepared a Mathematica file as supplemental material where all the solutions for this section are verified. Finally, in Appendix A, we recall the main tools we have used for our results. These tools are a solution with multiparameters of the Riccati system (A.1)–(A.6) and a modification of the transformation introduced in [59]; we have introduced an extra parameter $l_0 = \pm 1$ in order to use standard solutions for Peregrine-type soliton solutions. Also a 2D version of a generalized