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A kind of conditional connectivity of Cayley graphs generated by wheel graphs

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ABSTRACT

For a connected graph $G = (V, E)$, a subset $F \subseteq V$ is called an R^k-vertex-cut of G if $G - F$ is disconnected and each vertex in $V - F$ has at least k neighbors in $G - F$. The cardinality of the minimum R^k-vertex-cut is the R^k-vertex-connectivity of G and is denoted by $\kappa^k(G)$. The conditional connectivity is a measure to explore the structure of networks beyond the vertex-connectivity. Let $Sym(n)$ be the symmetric group on $\{1, 2, \dots, n\}$ and $\mathcal{T}$ be a set of transpositions of $Sym(n)$. Denote by $G(\mathcal{T})$ the graph with vertex set $\{1, 2, \dots, n\}$ and edge set $\{ij : (ij) \in \mathcal{T}\}$. If $G(\mathcal{T})$ is a wheel graph, then simply denote the Cayley graph $Cay(Sym(n), \mathcal{T})$ by WG_n. In this paper, we determine the values of κ^1 and κ^2 for Cayley graphs generated by wheel graphs and prove that $\kappa^1(WG_n) = 4n - 6$ and $\kappa^2(WG_n) = 8n - 18$.

1. Introduction

We begin with a brief section on terminology and notation and then motivate our results by a number of papers. Throughout this paper, we consider only undirected graphs with no loops and no multiple edges. Let $G = (V, E)$ be a finite graphs without loops and parallel edges. We follow [1,2,20] for notations and terminology not given here. Let $n = |V|$ and $m = |E|$ denote the order and size of G, respectively. The set of neighbors of a vertex u in G is denoted by $N_G(u)$, or briefly by $N(u)$; this set does not include u itself. For a subset $U \subseteq V$, we denote $N(U) = (\cup_{u \in U} N(u)) - U$. Sometimes, we use a graph itself to represent its vertex set, for instance, $N(H)$ means $N(V(H))$ if no confuse occurs, where H is a subgraph of G. The *degree* of a vertex u in G is the number $|N(u)|$ of edges at u, denoted by $d(u)$. The minimum vertex degree in a graph G is denoted by $\delta(G)$. A graph G is *connected* if for any two vertices there exists a path connecting them. A *vertex cut* of a graph G is a set $S \subseteq V$ such that $G - S$ has more than one component. The *vertex-connectivity* of G, denoted by $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A *disconnecting set* of edges is a set $F \subseteq E$ such that $G - F$ has more than one component. The *edge-connectivity* of G with at least two vertices, denoted by $\lambda(G)$, is the minimum size of disconnecting set.

Let Γ be a finite group and S be a subset of Γ such that the identity of the group does not belong to S. The *Cayley digraph*, denoted by $Cay(\Gamma, S)$, is the digraph with vertex set Γ and arc set $\{(g, g \cdot s): g \in \Gamma, s \in S\}$. We say that arc $(g, g \cdot s)$ has label s. In particular, if $S^{-1} = S$, then $Cay(\Gamma, S)$ is an undirected graph, which is called *Cayley graph*.

It is well known that the underlying topology of an interconnection network can be modeled by a graph G. One of the significant issues in studying a network is on how to measure its fault tolerance, among two major historical ones are the

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traditional vertex-connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$. It was shown in [25] that the higher these parameters are, the more reliable the network is. However, these parameters have some intrinsic shortcomings. The first one is that a lot of graphs with the same vertex-connectivity behave quite differently in fault tolerance. The second reason is that vertex-connectivity just measures the worst case failures, which seldom occur in the real world, the resilience of a network is drastically underestimated. To overcome the shortcomings, Esfahanian [10] proposed the concept of restricted connectivity, which was a measure of conditional fault tolerance for networks in the case of vertex failure and generalized by Latifi et al. in [14] and Oh and Choi in [18] independently to R^k-vertex-connectivity.

For a simple connected graph $G=(V,E)$, a subset $F\subseteq V$ is called an R^k-*vertex-set* of G if each vertex $u\in V-F$ has at least k neighbors in $G-F$, in other words, if the minimum degree δ of the survival graph $G-F$ satisfies $\delta(G-F)\geq k$. An R^k-*vertex-cut* of a connected graph G is an R^k-vertex-set F of G such that $G-F$ is disconnected. In network applications, we always call vertices in F the *faulty vertices*, while those in $V-F$ are said to be *good vertices*. The R^k-*vertex-connectivity* of G, denoted by $\kappa^k(G)$, is the cardinality of a minimum R^k-vertex-cut of G. We refer the readers to [24] for a concise introduction to this parameter, its impacts in the study of interconnection networks and references.

The idea behind the R^k-vertex-connectivity also lies in measuring fault tolerance of networks. Generally speaking, the probability that all failures concentrate around a vertex is often small. For example, n-dimensional hypercube Q_n and its variants form the basic class of interconnection networks, which possess 2^n vertices and vertex connectivity n. It is known that every minimum vertex-cut of Q_n is the set of neighbors of some vertex. Suppose n vertices fail in Q_n, then the probability that these n vertices form a vertex cut is $2^n/\binom{2^n}{n}$, which is very small when n is sufficient large. In the definition of R^k-vertex-set, the requirement that each good vertex has at least k good neighbors takes the above resilience into consideration.

It is fault tolerance that is concerned in this paper. The conditional connectivity concept is a measure to study the structure of networks beyond vertex-connectivity. Of course, it is not the unique measure. Other ideas include cyclic vertex connectivity [29], integrity [3], toughness [4] and strength (The strength of a network having non-negative edge weights, is the minimum over subsets A of edges, of the weight of A divided by the number of additional components created by deleting A) [7,8]. Another approach is to consider the structure of these networks after it is disconnected by deleting vertices. By this kind of structure, we focus on how badly the graph is disconnected. If the resulting graph has one big component as well as several small ones, then the core is still intact in the sense. Such a structural result is useful in the consideration of other connectivities. We note that the theme of fault tolerance with respect to certain properties is common, we encourage the interested readers to consult some recent papers [9,13,16,26] for more details.

Cayley graphs have a lot of properties which are desirable in an interconnection network [11,15]. It was known that vertex symmetry makes it possible to use the same routing protocols and communication schemes at all nodes; hierarchical structure facilitates recursive constructions; high fault tolerance implies robustness, among others.

In [10], Esfahanian calculated the exact κ^1-value for hypercube Q_n, which was generalized independently by Latifi et al. [14] and Oh and Choi [18] to the result: $\kappa^k(Q_n)=2n-2$ holds for $1\leq k\leq n$. In [12], Hu and Yang proved that $\kappa^1(S_n)=2n-4$, where S_n is the n-dimensional star graph.

In [6], Cheng et al. studied a class of Cayley graphs that generated by transpositions. They proved the following:

Theorem 1.1 ([6]). *Let G be the Cayley graph obtained from a generating graph A on $\{1,2,\dots,n\}$ with m edges, where $m\geq 7$. Suppose that T is a set of vertices of G such that $|T|\leq 2m-2$ if A has no triangles and $|T|\leq 2m-3$ if A has a triangle. Then one of the following is true:*

1. *$G-T$ is connected.*
2. *$G-T$ is disconnected with exactly two components, one of which is a singleton.*
3. *The generating graph A has no triangles, $G-T$ is disconnected with exactly two components, one of which is K_2, and $|T|=2m-2$.*
4. *The generating graph A has no triangles, $G-T$ is disconnected with exactly three components, two of which are singletons, and $|T|=2m-2$.*
5. *The generating graph A has a triangle, $G-T$ is disconnected with exactly three components, two of which are singletons, and $|T|=2m-3$.*

Next, we state another result in [6] which will be used in our proofs later.

Theorem 1.2 ([6]). *Let G be the Cayley graph obtained from a generating graph A on $\{1,2,\dots,n\}$ with m edges, where $n\geq 3$. Then G does not have the graph $K_{2,4}$ as a subgraph. If, in addition, A does not contain a triangle, then G does not have the graph $K_{2,3}$ as a subgraph.*

In 2010, Yang and co-authors determined the number κ^2 of Cayley graphs generated by transposition trees [27]. Zhang et al. [30] presented the exact value of κ^2 for n-dimensional alternating group graphs. In [5], Cheng et al. investigated a kind of conditional vertex connectivity of Cayley graphs generated by 2-trees, including the popular alternating group graphs. More recently, Wang et al. considered the κ^1 and κ^2 for the complete-transposition graphs [23].

The aim of this paper is to continue this programm and determine κ^1- and κ^2-values for Cayley graphs generated by wheel graphs. In the following section, we introduce the hierarchical structure of Cayley graphs generated by wheel graphs.