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The Crane equation $uu_{xx} = -2$: The general explicit solution and a case study of Chebyshev polynomial series for functions with weak endpoint singularities

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ABSTRACT

The boundary value problem $uu_{xx} = -2$ appears in Crane's theory of laminar convection from a point source. We show that the solution is real only when $|x| \leq \sqrt{\pi}/2$. On this interval, denoting the constants of integration by A and s, the general solution is $AV([x-s]/A)$ where the "Crane function" V is the parameter-free function $V = \exp\left(-\left\{\mathrm{erfinv}(-[2/\sqrt{\pi}])x\right\}^2\right)$ and erfinv(z) is the inverse of the error function. $V(x)$ is weakly singular at both endpoints; its Chebyshev polynomial coefficients a_n decrease proportionally to $1/n^3$. Exponential convergence can be restored by writing $V(x) = \sum_{n=0} a_{2n} T_{2n}(z[x])$ where the mapping is $z = \frac{\mathrm{arctanh}(x/\mho)}{L^2 + \left(\mathrm{arctanh}(x/\mho)\right)^2}$, $\mho = \sqrt{\pi}/2$. Another option is singular basis functions. $V \approx (1 - x^2/\mho^2)\left\{1 - 0.216 \log(1 - x^2/\mho^2)\right\}$ has a maximum pointwise error that is less 1/2000 of the maximum of the Crane function.

1. Introduction

Similarity solutions for axisymmetric convection from a point heating source has a lengthy history [1,13,15,17,33,34]. Crane derived an asymptotic series in powers of the Prandtl number and in the process was to forced to solve

$$uu_{xx} = -2 \tag{1}$$

Crane himself was able to solve his problem only through a slowly-convergent infinite series. In the next section, we derive an exact, explicit solution.

The general solution to a second order ODE must contain two constants of integration. We prove a trivial group invariance theorem that reduces the problem to a single parameter-free function $V(x)$ that we shall dub the "Crane function".

Fig. 1 shows the Crane function and its first two derivatives. The $x\log(x)$ singularities are evident in the steep slopes and curvatures near the endpoints.

The explicit solution involves the inverse of the error function, which we denote by erfinv(z). Because software to evaluate this is not readily available, a brief appendix on a never-failing algorithm to compute this function is included.

Another option is to compute the Crane function as a Chebyshev polynomial series. However, the Crane function has logarithmic singularities at both endpoints, slowing the rate of convergence to just third order. There are two ways to recover an exponential rate of convergence. The first is to use a change of coordinate ("mapping") as illustrated in Section 5.

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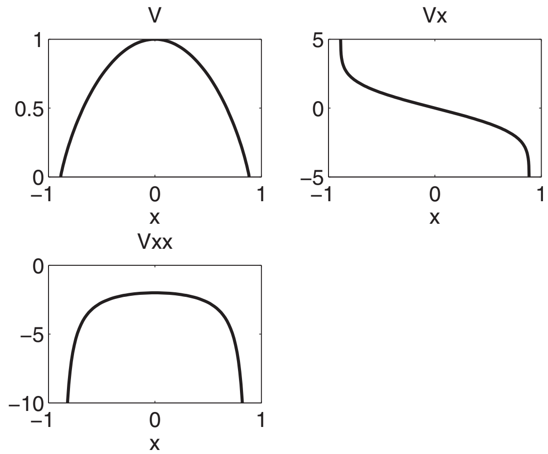

Fig. 1. Plot of the Crane function $V(x)$ and its first two derivatives. The function is real-valued only for $|x| \le \sqrt{\pi}/2 \approx 0.8862$. The function V is the unique solution to $V_{xx}V = -2,\ V_x(0) = 0,\ V(0) = 1$.

The other option is to use singular basis functions. This yields very accurate approximations at remarkably low order as shown in Section 6.

1.1. Background

The 1978 textbook by Bender and Orszag [2] discusses the Crane equation in Problem 4.26 on pg. 200. Part (a) asks the student to show that $u_{xx}u = 1$ subject to $u(0) = u(1) = 0$ has no real solutions. (Note that the curvature is opposite to our problem.) Part (b) asks for a proof that $u_{xx}u = -1$ subject to $u_x(0) = u(1) = 0$ has precisely two solutions that are nonsingular on the interval between the endpoints (the Crane function and its negative).

Their Fig. 4.11 gives a plot of the "exact solution" to $u_{xx}u = -1$, subject to $u(0) = u(1) = 0$, [for which no analytical solution is given] plus two local approximations proportional to $x\sqrt{-\log(x)}$ and $(1-x)\sqrt{-\log(1-x)}$. These forms are obtained on pg. 171 by trying $Ax(\log(-x)^b$ where A, b are constants, substituting into the differential equation, and matching leading powers.

Bender's best recollection, forty years later, is "that we just made it up out of thin air. I don't remember finding it in any previously published work. It is possible, of course, that Orszag found it in a paper, but if he did, I certainly don't remember his saying so." (Bender, private communication, 2015).'

More than half a century ago, Philip, FRS, wrote a review of properties of the inverse complementary error function [erfc] and its uses in nonlinear diffusion and porous flow problems [23].

To derive recurrences for derivatives of the inverse complementary error function, Philip found it expedient to introduce a function

$$B(\theta) \equiv \frac{2}{\sqrt{\pi}} \exp\left(-\left\{\mathrm{inverfc}\left(1 - [2/\sqrt{\pi}]x\right)\right\}^2\right), \qquad \theta \in [0, 2] \tag{2}$$

$$\equiv \frac{2}{\sqrt{\pi}} V\left(\frac{\sqrt{\pi}}{2}(\theta - 1)\right), \qquad \theta \in [0, 2] \tag{3}$$

which is thus a rescaled version of the Crane function. His paper is a treasure-house of identities and formulas for "erfcinv", his primary topic. He does not give a differential equation for $B(\theta)$, but does provide a table of its values. He notes that $\lim_{\theta \to 0} B(\theta)/\left\{2\theta\sqrt{-\theta}\right\} = 1$.

2. Invariance and explicit solutions

2.1. Group invariance

Theorem 1. *Let $u(x)$ denote a solution to*

$$u_{xx}u = -2 \tag{4}$$

1. *If $u(x)$ is a solution, then $w(x) \equiv v(x - s)$ is also a solution for any constant s. [Translational Invariance]*
2. *If If $u(x)$ is a solution, then*