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Reconstruction of time-dependent coefficients from heat moments

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A B S T R A C T

This paper investigates the inverse problems of simultaneous reconstruction of time-dependent thermal conductivity, convection or absorption coefficients in the parabolic heat equation governing transient heat and bio-heat thermal processes. Using initial and boundary conditions, as well as heat moments as over-determination conditions ensure that these inverse problems have a unique solution. However, the problems are still ill-posed since small errors in the input data cause large errors in the output solution. To overcome this instability we employ the Tikhonov regularization. A discussion of the choice of multiple regularization parameters is provided. The finite-difference method with the Crank–Nicolson scheme is employed as a direct solver. The resulting inverse problems are recast as nonlinear minimization problems and are solved using the *lsqnonlin* routine from the MATLAB toolbox. Numerical results are presented and discussed.

1. Introduction

Simultaneous determination of several unknown physical property coefficients in heat transfer which dependent on time, space or temperature has been investigated in various studies, see e.g. [6–9,11].

In a recent paper [6] by the authors we have investigated the inverse problems of simultaneous numerical reconstruction of time-dependent thermal conductivity and convection coefficients in a one-dimensional parabolic equation from Cauchy boundary data measurements represented by the boundary temperature and heat flux. In this paper, we investigate the reconstruction of the same coefficients, as well as of the absorption coefficient, using the measurement of the heat moments instead of the heat flux.

The paper is organized as follows: in Section 2, the mathematical of formulation of the inverse problems are reformulated and uniqueness results are stated. In Section 3, the numerical solution of the direct problem based on finite difference method with the Crank–Nicolson scheme is presented. In Section 4, the numerical approach to solve the minimization of the nonlinear Tikhonov regularization functional is presented. The numerical results for various examples are presented and discussed in Section 5. The choice of multiple regularization parameters is also addressed. Finally, conclusions are presented in Section 6.

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2. Mathematical formulation

Fix the parameters $L > 0$ and $T > 0$ representing the length of a finite slab and the time duration, respectively. Denote by $\Omega_T := (0, L) \times (0, T)$ the solution domain. We consider the parabolic heat equation

$$\frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t)\frac{\partial u}{\partial x}(x,t) + c(t)u(x,t) + f(x,t), \quad (x,t) \in \Omega_T, \tag{1}$$

where $a > 0$, b, c and f are coefficients, and $u(x, t)$ is the temperature. For simplicity, we have assumed that the heat capacity is constant and taken to be unity. Eq. (1) has to be solved subject to the initial condition

$$u(x,0) = \phi(x), \quad 0 \le x \le L, \tag{2}$$

and the Dirichlet boundary conditions

$$u(0,t) = \mu_1(t), \quad u(L,t) = \mu_2(t), \quad 0 \le t \le T. \tag{3}$$

If a, b, c and f are given then (1)–(3) constitute a direct Dirichlet problem for the temperature $u(x, t)$. Other outputs of interest are the heat fluxes

$$-a(t)u_x(0,t) =: q_0(t), \quad a(t)u_x(L,t) =: q_L(t), \quad 0 \le t \le T, \tag{4}$$

and the heat moments

$$H_k(t) = \int_0^L x^k u(x,t)dx, \quad k = 0, 1, \quad 0 \le t \le T. \tag{5}$$

However, if any of the coefficients a, b, c and/or f are not known then we are dealing with inverse coefficient identification problems.

Prior to this study, the simultaneous identification of the coefficients $a(t)$ and $b(t)$ in the problem (1)–(3) with the additional flux data (4) has been considered in [6]. In this paper, we consider the simultaneous reconstruction of the same time-dependent coefficients, as well as of $a(t)$ and $c(t)$, but from the heat moments (5) instead of the heat fluxes (4). The uniqueness of solution of these inverse problems is stated in the next two sections.

2.1. Inverse problem 1

Assuming that $c(t) = 0$, the inverse problem 1 (IP1) requires the simultaneous determination of the time-dependent thermal conductivity $a(t) > 0$ and the convection (or advection) coefficient $b(t)$, together with the temperature $u(x, t)$ satisfying

$$\frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t)\frac{\partial u}{\partial x}(x,t) + f(x,t), \quad (x,t) \in \Omega_T, \tag{6}$$

subject to (2), (3) and (5).

The uniqueness of solution $(a(t), b(t), u(x, t))$ of this inverse problem was established in [13] and reads as follows.

Theorem 1. *Let $\phi \in C^1[0, L]$, $\mu_k \in C^1[0, T]$, $H_k \in C^1[0, T]$ for $k = 0, 1$, and $f \in C(\overline{\Omega_T})$.*
Suppose that the following condition is satisfied:

$$U_1(t) := (\mu_2(t) - \mu_1(t))\int_0^L xf(x,t)dx - (L\mu_2(t) - H_0(t))\int_0^L f(x,t)dx \neq 0, \quad \forall t \in [0,T]. \tag{7}$$

Then a solution $(a(t), b(t), u(x,t)) \in C[0,T] \times C[0,T] \times (C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T}))$ with $a(t) > 0$ for $t \in [0, T]$, to the problem (2), (3), (5) and (6) is unique.

Remark 1. Observe that by multiplying Eq. (6) by x^k, $k = 0, 1$, integrating with respect to x from 0 to L, and taking into account condition (5), we obtain,

$$H_0'(t) = a(t)(u_x(L,t) - u_x(0,t)) + b(t)(u(L,t) - u(0,t)) + \int_0^L f(x,t)dx,$$

$$H_1'(t) = a(t)(Lu_x(L,t) - u(L,t) + u(0,t)) + b(t)(Lu(L,t) - H_0(t)) + \int_0^L xf(x,t)dx.$$

Taking $t = 0$ in these equations, using the compatibility conditions $u_x(0,t) = \phi'(0)$, $u_x(L,t) = \phi'(L)$, the Dirichlet boundary conditions (3) and solving for $a(0)$ and $b(0)$, we obtain,

$$a(0) = (\Delta(0))^{-1}\Bigg[\left(H_0'(0) - \int_0^L f(x,0)dx\right)(L\mu_2(L) - H_0(0)) - (\mu_2(L) - \mu_1(0))\left(H_1'(0) - \int_0^L xf(x,0)dx\right)\Bigg],$$