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A new iterative method for solving complex symmetric linear systems

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A B S T R A C T

Based on implementation of the quasi-minimal residual (QMR) and biconjugate *A*-orthogonal residual (BiCOR) method, a new Krylov subspace method is presented for solving complex symmetric linear systems. The new method can be combined with arbitrary symmetric preconditioners. The preconditioned modified Hermitian and Skew-Hermitian splitting (PMHSS) preconditioner is used to accelerate the convergence rate of this method. Numerical experiments indicate that the proposed method and its preconditioned version are efficient and robust, in comparison with other Krylov subspace methods.

1. Introduction

When partial differential equations such as the Helmholtz and Maxwell equations that involve complex coefficient functions or complex boundary conditions are discretized, we need to solve large linear systems

$$Ax = b, \tag{1}$$

where A is an $N \times N$ complex symmetric nonsingular matrix and $x, b \in \mathbb{C}^N$. Complex symmetric linear systems also arise from other various physical problems, such as electromagnetic scattering problem [1], molecular scattering [2], structural dynamics [3] and quantum mechanics [4]. When huge memory is a concern, the direct method is no longer practical to employ, and then Krylov subspace methods are considered as one class of the important and efficient techniques. van der Vorst and Mellissen [5] developed the conjugate orthogonal conjugate gradient (COCG) method based on conjugate orthogonal relation, which needs only about half the amount of work per iteration as the biconjugate gradient (BiCG) method [6]. Freund [7] proposed a conjugate gradient-type method with quasi-minimal residual (QMR) [8] based on the Lanczos recursion. Bunse-Gerstner and Stöver [9] developed another conjugate gradient-type method, which is based on unitary equivalence transformations of complex symmetric coefficient matrix A to symmetric tridiagonal form. By extending the conjugate residual method [10] to nonsymmetric linear systems, Sogabe et al. [11] presented the BiCR method that possesses smoother convergence behavior compared to the BiCG method for real nonsymmetric linear systems. By applying the BiCR method to complex symmetric linear systems, Sogabe and Zhang [12] derived the conjugate orthogonal conjugate residual (COCR) method which exhibits smoother convergence behavior than the COCG method.

The SYMMLQ and MINRES methods [13] are the standard conjugate gradient-type Krylov subspace methods for solving symmetric indefinite linear systems. However, when preconditioning is implemented, the preconditioner need to be

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symmetric positive definite, which limits the choice of possible preconditioners for SYMMLQ and MINRES methods when the coefficient matrix is highly indefinite. To remedy this difficult, Freund and Nachtigal [14] presented a symmetric quasi-minimal residual (SQMR) method that can be combined with arbitrary symmetric preconditioner. Note that the SQMR method is derived from the coupled two-term QMR algorithm [15,16] and the BiCG method. Recently, Jing et al. [17] and Carpentieri et al. [18] explored a biconjugate A-orthogonal residual (BiCOR) method based on biconjugate A-orthonormalization process. To enhance stability and convergence rate of the BiCOR method, some product-type iterative methods, including CORS [17,18], BiCORSTAB [17,18], BiCORSTAB2 [19], GPBiCOR [20], QMRCORSTAB [21] and GCORS [22] methods, are designed. Numerical results from scientific and engineering problems show that these methods are competitive with or superior to other Krylov subspace methods. Inspired by the advantage of the BiCOR family methods and the SQMR method, we derive a new iterative method for solving complex symmetric linear system by using the connection of the QMR method and the BiCOR method. We abbreviate the resulting method to the SQMOR method. The new method can also be combined with arbitraly symmetric preconditioners and competitive with or superior to the SQMR method in many cases, especially highly indefinite complex symmetric linear systems.

Bai et al. [23,24] presented the Hermitian and Skew-Hermitian splitting (HSS) method and its modification (called MHSS method) for solving non-Hermitian positive definite linear systems, and proved that the HSS method and the MHSS method converge unconditionally. To accelerate convergence rate of the MHSS iteration method, Bai et al. [25] developed the preconditioned MHSS (PMHSS) method. Numerical experiments demonstrate that the PMHSS method and the GMRES method with the PMHSS preconditioner embody meshsize-independent and parameter-insensitive convergence behavior. To improve the performance of the SQMOR method, we can employ the PMHSS preconditioner. Numerical experiments show that the SQMOR method with the PMHSS preconditioner converges in considerably fewer iterations as compared with other preconditioners, such as symmetric successive overrelaxation (SSOR) [26] preconditioner.

The remainder of the paper is organized as follows. In Section 2, we describe an implementation of the QMR method based on the BiCOR method and derive the QMRBiCOR method by using connection of the QMR method and the BiCOR method. In Section 3, we propose the SQMOR method and describe the SQMOR method with the PMHSS preconditioner. Numerical experiments are given in Section 4 to show the effectiveness of the SQMOR method and the preconditioned SQMOR method. Finally, we make some conclusions and remarks in Section 5.

Throughout this paper, we use the follow notations. Let the overbar "–" denote the conjugate complex of a scalar, vector or matrix, Z^T and Z^H denote the transpose and the conjugate transpose of a vector or matrix Z, respectively. The Krylov subspace $\mathcal{K}_m(A, v)$ generated by a matrix $A \in \mathbb{C}^{N\times N}$ and a vector $v \in \mathbb{C}^N$ is defined as $\mathcal{K}_m(A, v) = span(v, Av, \ldots, A^{m-1}v)$.

2. The QMRBiCOR method

The BiCOR method often converges faster and smoother than the BiCG [6] method in many practical problems, but it still shows an irregular convergence behavior in the residual norm. For convenience, the BiCOR [17,18] method is described as in Algorithm 1.

Algorithm 1 The BiCOR method.

Compute $r_0 = b - Ax_0$ for some initial guess x_0.
Choose $r_0^* = p(A)r_0$ such that $(r_0^*, Ar_0) \neq 0$, where $p(t)$ is a polynomial in t (for example, $r_0^* = Ar_0$).
Set $p_0 = r_0$, $p_0^* = r_0^*$, $q_0 = Ap_0$, $q_0^* = A^H p_0^*$, $\widehat{r}_0 = Ar_0$, $\rho_0 = (r_0^*, \widehat{r}_0)$.
for $j = 1, 2, \ldots$ **do**
  $\sigma_j = (q_j^*, q_j)$
  $\alpha_j = \frac{\rho_j}{\sigma_j}$
  $x_{j+1} = x_j + \alpha_j p_j$
  $r_{j+1} = r_j - \alpha_j q_j$
  $x_{j+1}^* = x_j^* + \overline{\alpha}_j p_j^*$
  $r_{j+1}^* = r_j^* - \overline{\alpha}_j q_j^*$
  $\widehat{r}_{j+1} = Ar_{j+1}$
  $\rho_{j+1} = (r_{j+1}^*, \widehat{r}_{j+1})$
  $\beta_{j+1} = \frac{\rho_{j+1}}{\rho_j}$
  $p_{j+1} = r_{j+1} + \beta_{j+1} p_j$
  $p_{j+1}^* = r_{j+1}^* + \overline{\beta}_{j+1} p_j$
  $q_{j+1} = \widehat{r}_{j+1} + \beta_{j+1} q_j$
  $q_{j+1}^* = A^H p_{j+1}^*$
end for