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An efficient two-step algorithm for the stationary incompressible magnetohydrodynamic equations☆

Jilian Wu, Demin Liu, Xinlong Feng*, Pengzhan Huang

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, PR China

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ABSTRACT

A new highly efficient two-step algorithm for the stationary incompressible magnetohydrodynamic equations is studied in this paper. The algorithm uses a lower order finite element pair (i.e., $P_1b - P_1 - P_1$) to compute an initial approximation, that is using the Mini-element (i.e., $P_1b - P_1$) to approximate the velocity and pressure and P_1 element to approximate the magnetic field, then applies a higher order finite element pair (i.e., $P_2 - P_1 - P_2$) to solve a linear system on the same mesh. Furthermore, the convergence analyses of standard Galerkin finite element method and the two-step algorithm are addressed. Lastly, numerical experiments are presented to verify both the theory and the efficiency of the algorithm.

1. Introduction

Magnetohydrodynamic (MHD) equations which have been widely used in industry and engineering, such as liquid metal cooling of nuclear reactors, process metallurgy, simulate aluminum electrolysis cells and so on are composed of Navier–Stokes equations of fluid dynamics and Maxwells equations of electromagnetism. These equations are used to describe the interaction between a viscous, incompressible, electrically conducting fluid and an external magnetic field. More comprehensive understanding of the physical background of MHD equations can be found in [1]. In this paper, we shall use an efficient two-step algorithm to solve the stationary incompressible MHD equations:

$$\begin{cases} -R_e^{-1}\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_c\text{curl}\mathbf{B}\times\mathbf{B} = \mathbf{f} & \text{in } \Omega,\\ S_cR_m^{-1}\,\text{curl}(\text{curl}\mathbf{B}) - S_c\text{curl}(\mathbf{u}\times\mathbf{B}) = \mathbf{g} & \text{in } \Omega,\\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega,\\ \nabla\cdot\mathbf{B} = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

with the following boundary conditions

$$\begin{cases} \mathbf{u}|_{\partial\Omega} = 0 & \text{(no slip condition)},\\ (\mathbf{B}\cdot\mathbf{n})|_{\partial\Omega} = 0, \quad (\mathbf{n}\times\text{curl}\mathbf{B})|_{\partial\Omega} = 0 & \text{(perfectly conducting wall)}, \end{cases} \tag{2}$$

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* Corresponding author. Tel.: 869918582561.

E-mail addresses: happy_whw@163.com (J. Wu), followtime@126.com (D. Liu), fxlmath@xju.edu.cn, fxlmath@gmail.com (X. Feng), hpzh007@yahoo.com (P. Huang).

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where Ω is a bounded convex polygonal/polyhedral domain in $\mathbb{R}^d (d = 2, 3)$, $\mathbf{u}$ is velocity, $\mathbf{B}$ is magnetic field, p is pressure, $\mathbf{n}$ is outer unit normal of $\partial\Omega$, and the three parameters R_e, R_m, S_c represent hydrodynamic Reynolds number, magnetic Reynolds number and coupling number, respectively. And the known functions of $\mathbf{f}$ and $\mathbf{g}$ are source terms, with $\mathbf{g}$ being solenoidal. Here $\mathbf{u} = (u_1, u_2, 0), \mathbf{B} = (B_1, B_2, 0), \mathbf{f} = (f_1, f_2, 0)$, and $\mathbf{g} = (g_1, g_2, 0)$ for $d = 2$ ($\mathbf{u} = (u_1, u_2, u_3), \mathbf{B} = (B_1, B_2, B_3), \mathbf{f} = (f_1, f_2, f_3)$, and $\mathbf{g} = (g_1, g_2, g_3)$ for $d = 3$).

Because MHD equations have practical significance in many fields of production and life, many researchers have gave some research results for this problem (see [2,4–11,25] and the references therein) in the past decades. Among the above results, we only list some of them in the following. Gunzburger et al. [2] gave a finite element (FE) method and its numerical analysis for stationary incompressible MHDs. Layton and his collaborators [4,5] offered a two-level method for the incompressible fluid flows, that is solving the nonlinear problem on a coarse mesh and then solving a linear one on a fine mesh based on Newton iterative algorithm. In [6–8], the mixed and stabilized FE methods were used to solve MHD equations. Additionally, He [10] studied an unconditional convergence of the Euler semi-implicit scheme and Yuksel and Ingram [9] investigated a full discretization of Crank–Nicolson scheme for the non-stationary MHD equations with small magnetic Reynolds numbers.

In addition, because MHD equations include not only the strong nonlinearity but also the coupling between Navier–Stokes equation and Maxwell equations, development of an efficient computational algorithm for this problem has practical significance and has drawn the attention of many researchers nowadays. Recently, two-level method has been considered to solve many nonlinear problems (see [12–17,22]). As its basic idea is solving one nonlinear system on a coarse mesh as an iterative initial value approximation of fine mesh and then solving one linear system on a fine mesh. That can save a large amount of computation compared with the one-level methods. Based on two-level method, Huang et al. [18] and Wu et al. [19] gave an efficient two-step algorithm for the Navier–Stokes problem and steady-state natural convection problem, respectively.

In this paper, we shall use the new two-step algorithm to solve stationary incompressible MHD equations and give the corresponding numerical analysis and numerical tests. That is using $P_1 b - P_1 - P_1$ to solve a nonlinear system and then using $P_2 - P_1 - P_2$ to solve a linear system. The basic idea of our two-step method is to compute an initial approximation based on a lower order element, then to solve a linear system based on a higher order element. But the method only needs one mesh size, which can avoid the discussion on the relation of the coarse and fine meshes, and which is different from the two-level method (see [18,19]). Besides, we guarantee magnetic field $\mathbf{B}$ weakly divergence-free by introducing divergence-free subspaces ($\mathbf{V}$ and $\mathbf{V}_n$) which are proposed by Gunzburger et al. [2] and Dong et al. [11].

The remainder of this paper is as follows. In Section 2, we recall some basic notations and results of problem (1) and (2). In Section 3, we present standard Galerkin FEMs using a lower order element and a higher order element and give the corresponding error estimates for stationary incompressible MHD problem, respectively. In the next section, we present a new highly efficient two-step algorithm with optimal-order error estimates. Then, in Section 5, numerical experiments show that our method is efficient and reliable. Finally, we end with a short conclusion in Section 6.

2. Preliminaries

In this section, we will construct the variable formulation for problem (1) and give some necessary assumptions which will be frequently used in the following sections. Firstly, we introduce the standard scalar Sobolev space $H^m(\Omega) = W^{m,2}(\Omega)$ for nonnegative integer m with norm $\|v\|_m = (\sum_{|\gamma|\le m} \|D^\gamma v\|_0^2)^{1/2}$ and semi-norm $|v|_m = (\sum_{|\gamma|=m} \|D^\gamma v\|_0^2)^{1/2}$, and for vector-value functions, we use the Sobolev space $\mathbf{H}^m(\Omega) = (H^m(\Omega))^d$ with norm $|\mathbf{v}|_m = (\sum_{i=1}^d \|v_i\|_m^2)^{1/2}$ (see [11]). Then we introduce the following particular subspaces of $\mathbf{H}^1(\Omega)$ that satisfy specific boundary conditions (see [2,10,11]):

$$\mathbf{X} := \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\},$$
$$\mathbf{W} := \mathbf{H}_n^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\},$$

and subspaces of (weakly) divergence-free functions:

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X} : \nabla \cdot \mathbf{v} = 0, \quad \text{in} \quad \Omega\},$$
$$\mathbf{V}_n = \{\mathbf{v} \in \mathbf{W} : \nabla \cdot \mathbf{v} = 0, \quad \text{in} \quad \Omega\},$$

and the subspace of $L_2(\Omega)$:

$$Q := L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\right\}.$$

Here and below, the space $L^2(\Omega)$ is equipped with L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. Further, we employ the product space $\mathbf{W}_{0n}(\Omega) = \mathbf{H}_0^1(\Omega) \times \mathbf{H}_n^1(\Omega)$ with the usual graph norm $\|(\mathbf{v}, \Phi)\|_1$, where $\|(\mathbf{v}, \Phi)\|_i = (\|\mathbf{v}\|_i^2 + \|\Phi\|_i^2)^{1/2}$ for all $\mathbf{v} \in \mathbf{H}^i(\Omega) \cap \mathbf{X}$, $\Phi \in \mathbf{H}^i(\Omega) \cap \mathbf{W} (i = 1, 2)$. And the dual space $\mathbf{H}^{-1}(\Omega)$ of $\mathbf{H}_0^1(\Omega)$ with the following norm:

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega), \mathbf{v} \neq 0} \frac{\langle \mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_1},$$

where $\langle \cdot,\cdot \rangle$ denotes duality product between the function space $\mathbf{H}_0^1(\Omega)$ and its dual.