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Some Uzawa-type finite element iterative methods for the steady incompressible magnetohydrodynamic equations☆

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A B S T R A C T

In this paper, we apply some Uzawa-type iterative algorithms to the steady magnetohydrodynamic (MHD) equations discretized by mixed finite element method (FEM). Different from other iterative methods, they correct pressure at each iteration and converge geometrically with a contract number which is independent of mesh size h and iterative step n. Finally, the computational performance of our proposed methods are investigated by a numerical experiment.

1. Introduction

In this paper, we consider the following steady incompressible MHD equations [1]:

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - R_e^{-1}\triangle\mathbf{u} + S_c\mathbf{H}\times\mathrm{curl}\mathbf{H} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \mathrm{div}\mathbf{u} = 0, & \text{in } \Omega, \\ S_cR_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{H}) - S_c\mathrm{curl}(\mathbf{u}\times\mathbf{H}) = \mathbf{g}, & \text{in } \Omega, \\ \mathrm{div}\mathbf{H} = 0, & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Here, $\mathbf{u}$ denotes the velocity field, $\mathbf{H}$ the magnetic field, $\mathbf{f}$ and $\mathbf{g}$ the external force terms, $p = p(x)$ the pressure, $x \in \Omega$, R_e the hydrodynamic Reynolds number, R_m the magnetic Reynolds number, S_c the coupling number, and the density of the fluid is assumed to be 1. Correspondingly, the functions $\mathbf{u}$, $\mathbf{H}$, $\mathbf{f}$ and $\mathbf{g}$ can be described by:

$$\mathbf{u} = (u_1(x), u_2(x), 0), \quad \mathbf{H} = (H_1(x), H_2(x), 0), \quad \mathbf{f} = (f_1(x), f_2(x), 0), \quad \mathbf{g} = (g_1(x), g_2(x), 0),$$

for $d = 2$, and

$$\mathbf{u} = (u_1(x), u_2(x), u_3(x)), \quad \mathbf{H} = (H_1(x), H_2(x), H_3(x)),$$
$$\mathbf{f} = (f_1(x), f_2(x), f_3(x)), \quad \mathbf{g} = (g_1(x), g_2(x), g_3(x)),$$

for $d = 3$.

In this paper the system (1) is considered in conjunction with the following boundary conditions:

$$\begin{cases} \mathbf{u}|_{\partial\Omega} = 0, \quad \text{(no-slip condition)}, \\ \mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{n}\times\mathrm{curl}\mathbf{H}|_{\partial\Omega} = 0, \quad \text{(perfectly wall)}, \end{cases} \tag{2}$$

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where $\mathbf{n}$ is the outer unit normal of $\partial\Omega$.

In recent decades, lots of numerical approaches are devoted to MHD equations. Gunzburger et al. (1991) firstly discussed the mixed FEM for steady MHD equations in convex polyhedral or with a C^2 boundary [1]. For nonconvex polyhedral domain or re-entrant corners, Schötzau (2004) proposed to use Nedelec finite elements instead of Lagrange finite elements to approximate the magnetic field for steady MHD equations [2]. Hasler et al. (2004) adopted Lagrange finite elements by using a weight regularization technique [3]. To avoid the inf-sup condition, Gerbeau et al. (2006) presented a stabilized FEM for incompressible MHD equations [4]. Codina et al. (2006) considered a stabilized finite element method for stationary MHD equations based on a simple algebraic version of the subgrid scale variational concept [5]. Presently, Planas et al. (2011) studied the inductionless MHD problem using a stabilized finite element method [6] and Dong et al. (2014) analyzed the convergence of the three finite element iterative methods for steady MHD equations [7]. And there are many studies about the MHD equations [8–16].

It is well known that velocity $\mathbf{u}$ and the pressure p in (1) are coupled together by the incompressibility constraint div $\mathbf{u} = 0$, which makes the system difficult to solve numerically. A popular strategy to overcome this difficulty is to relax the incompressibility constraint in an approximate way, resulting in a class of pseudocompressibility methods, among which are the Uzawa method, penalty method, the pressure stabilization method and the artificial compressibility method. We mainly consider the Uzawa method in this work. In [17], Temam proposed an Uzawa method for solving steady Stokes and Navier–Stokes equations. Then Nochetto et al. (2004) considered optimal relaxation parameter for the Uzawa method of the Stokes equations [18]. Currently, Chen et al. (2014) studied some Uzawa methods for steady Navier–Stokes equations [19].

The concern of this paper is to apply decoupling idea to the steady incompressible MHD equations discretized by mixed finite element method. First, we propose a nonlinear Uzawa-type iterative scheme to decouple the constraint system. Besides, we give a linearized inner iterative scheme, which uses inner iteration to solve a nonlinear equation in the above nonlinear scheme. Motivated by [19] and [7], furthermore, a linearized Uzawa-type iterative scheme is given, which linearizes the nonlinear term. Besides, we analyze their convergence rate. Finally, Some other possible decoupled schemes are showed.

The outline of this paper is as follows. In Section 2, functional setting of the stationary MHD equations is given. In Section 3, some basic results are given. Section 4 is devoted to the three Uzawa-type iterative schemes and their convergence. Section 5 is reported to show numerical performance and accuracy of our algorithms.

2. Functional setting of the problem

To write the variational form of problem (1)-(2), we introduce the Sobolev spaces $W^{k,r}(\Omega)$ for all non-negative integers k and r equipped with the standard Sobolev norms $\|\phi\|_{k,r}$ [20]. In particular, we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$ when $r = 2$. Vector-valued functions will be denoted in boldface, that is, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. For convenience, we set

$$\begin{aligned}
X &:= (H_0^1(\Omega))^d = \{\mathbf{w} \in (H^1(\Omega))^d : \mathbf{w}|_{\partial\Omega} = 0\},\\
W &:= (H_n^1(\Omega))^d = \{\mathbf{w} \in (H^1(\Omega))^d : \mathbf{w}\cdot\mathbf{n}|_{\partial\Omega} = 0\},\\
M &:= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q d\mathbf{x} = 0\}.
\end{aligned}$$

We use the equivalent norm $\|\mathbf{w}\|_1 = \|\nabla\mathbf{w}\|_0$ of X, and employ the product space $W_{0n} = (H_0^1(\Omega))^d \times (H_n^1(\Omega))^d$ with the usual graph norm $\|(\mathbf{v},\mathbf{B})\|_1$, where $\|(\mathbf{v},\mathbf{B})\|_i = (\|\mathbf{v}\|_i^2 + \|\mathbf{B}\|_i^2)^{1/2}$ for all $\mathbf{v} \in (H^i(\Omega))^d \bigcap X, \mathbf{B} \in (H^i(\Omega))^d \bigcap W$ $(i = 0, 1, 2)$. $(H^{-1}(\Omega))^d$ denotes the dual of $(H_0^1(\Omega))^d$ with the norm:

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in(H_0^1(\Omega))^d, \mathbf{v}\neq 0} \frac{\langle \mathbf{f}, \mathbf{v}\rangle}{\|\mathbf{v}\|_1},$$

where $\langle\ \cdot, \cdot\rangle$ denotes duality product between the function space $(H_0^1(\Omega))^d$ and its dual.

In addition, we give some formulations used frequently later on [1]

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\cdot\mathbf{d} = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = -(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{d}\times\mathbf{c}), \tag{3}$$

and

$$\int_\Omega \text{curl}\Phi\cdot\mathbf{H}d\mathbf{x} = -\int_{\partial\Omega}(\Phi\times\mathbf{n})\cdot\mathbf{H}d\mathbf{s} + \int_\Omega(\Phi\cdot\text{curl}\mathbf{H})d\mathbf{x}, \tag{4}$$

$$\begin{aligned}
(\text{curl}(\mathbf{w}\times\Phi), \mathbf{H}) &= -\langle(\mathbf{w}\times\Phi)\times\mathbf{n}, \mathbf{H}\rangle|_{\partial\Omega} + (\mathbf{w}\times\Phi, \text{curl}\mathbf{H}) = (\mathbf{w}\times\Phi, \text{curl}\mathbf{H})\\
&= -(\text{curl}\mathbf{H}\times\Phi, \mathbf{w}), \quad \forall \mathbf{w}\in X, \Phi, \mathbf{H}\in W.
\end{aligned}$$

Define the following forms:

$$\begin{aligned}
A_0((\mathbf{v},\mathbf{B}), (\mathbf{w},\Phi)) &= a_0(\mathbf{v},\mathbf{w}) + b_0(\mathbf{B},\Phi),\\
a_0(\mathbf{v},\mathbf{w}) &= R_e^{-1}(\nabla\mathbf{v}, \nabla\mathbf{w}),
\end{aligned}$$