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New local generalized shift-splitting preconditioners for saddle point problems[☆]

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ABSTRACT

Based on a local generalized shift-splitting of the saddle point matrix with symmetric positive definite (1, 1)-block and symmetric positive semidefinite (2, 2)-block, a new local generalized shift-splitting preconditioner with two shift parameters for solving saddle point problems is proposed. The preconditioner is extracted from a new local generalized shift-splitting iteration and can lead to the unconditional convergence of the iteration. In addition, we consider solving the saddle point systems by preconditioned Krylov subspace methods and discuss some properties of the preconditioned saddle point matrix with a deteriorated preconditioner, such as eigenvalues, eigenvectors, and degree of the minimal polynomial. Numerical experiments arising from a finite element discretization model of the Stokes problem are given to validate the effectiveness of the proposed preconditioner.

1. Introduction

We consider the solution of the following saddle point linear system

$$\mathcal{A}\mathbf{u} \equiv \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv \mathbf{b}. \tag{1}$$

In this linear system, $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (SPD) matrix, $C \in \mathbb{R}^{m\times m}$ is a symmetric positive semidefinite matrix, $B \in \mathbb{R}^{m\times n}$ with $m \le n$ has full row rank. B^T denotes the transpose of B. Moreover, $x, f \in \mathbb{R}^n$ and $y, g \in \mathbb{R}^m$. With these assumptions, the coefficient matrix (or the saddle point matrix in another word) $\mathcal{A}$ is nonsingular and the solution of (1) uniquely exists [1]. The saddle linear systems arise from a number of scientific computing and engineering applications [1,5,16,17,32], such as constrained optimization, optimal control, computational fluid dynamics, numerical solution of the Navier–Stokes equations by mixed finite element discretization, and so on. So it has received much of interest from mathematical reseachers in recent decades.

In general, the matrices A, B, C in $\mathcal{A}$ are large and sparse. The solution of (1) needs to be reached by iterative methods, such as stationary iterative methods and Krylov subspace methods. A variety of stationary iterative methods can be found in the literatures [6–9,20,21,37,38,45], including the Uzawa-type schemes, the Hermitian and skew-Hermitian splitting (HSS)

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methods, etc. For Krylov subspace methods we refer to [1–3,43,44] and the references therein, including minimum residual method (MINRES) and generalized minimal residual method (GMRES) with different preconditioners, as well as iterative projection methods [35], iterative null space methods [1], and so on. Both stationary iterative methods and Krylov subspace methods have their own advantages, as stated in [23], the former methods require less computer memory, while the later methods are more efficient in practical implementation if effective preconditioners are employed to guarantee their rapid convergence rates. There have been active discussions on preconditioners for saddle point problem. Recently, Bai [18] briefly reviewed the direct methods and iteration methods including the Krylov subspace iteration methods for solving linear system, and stated the advantages and disadvantages among these methods, which are fit for the saddle point problems.

To improve the convergence rates, different kinds of preconditioners have been proposed by matrix splitting or matrix factorization [1,11,14,19,21,22,30,34,39]. It is worthy to mention that in [10], Bai et al. presented a shift-splitting preconditioner for non-Hermitian positive definite linear systems, which arises from the splitting of the coefficient matrix. Subsequently, theoretical properties of two inexact Hermitian/skew-Hermitian splitting (IHSS) iteration methods and inexact implementations of the proposed preconditioner were discussed in [13] and the references therein. Cao et al. [23] generalized the idea in [10]. Based on a shift-splitting of the saddle point matrix $\mathcal{A}$ with $C = 0$, they presented a shift-splitting preconditioner and a local shift-splitting preconditioner for saddle point problems, and studied some properties of the local shift-splitting preconditioned matrix. The preconditioners contained only one shift parameter α. In 2014, Cao et al. [24] originally proposed a generalized shift-splitting (GSS) preconditioner with two shift parameters to solve nonsingular saddle point problems, and proved the unconditional convergent property of the corresponding iteration method. Later, Cao and Miao [26] extended the GSS iteration method to solve singular nonsymmetric saddle point problems. Also, the generalized shift-splitting preconditioner was extended to solve the saddle point problems with symmetric positive definite (1, 1)-block and symmetric positive semidefinite (2, 2)-block [41], and the nonsingular saddle point problems with nonsymmetric positive definite (1, 1)-block [25] and the singular saddle point problems with symmetric positive definite (1, 1)-block [28].

In this paper, we focus on describing a new local generalized shift-splitting preconditioner with two shift parameters for solving saddle point problems. The saddle point matrix is with symmetric positive definite (1, 1)-block and symmetric positive semidefinite (2, 2)-block. The preconditioner is extracted from a new local generalized shift-splitting iteration and can lead to the unconditional convergence of the iteration. In addition, we consider solving the saddle point systems by preconditioned Krylov subspace methods and discuss the spectral properties of the preconditioned saddle point matrices with a deteriorated preconditioner. Numerical experiments arising from a finite element discretization model of the Stokes problem are given to validate the effectiveness of the proposed preconditioners.

2. The new local generalized shift-splitting preconditioner

By introducing two positive parameters α and β, we construct two block matrices

$$\mathcal{M}_{\alpha,\beta} = \frac{1}{2}\begin{bmatrix} \alpha H_1 + A & B^T \\ -B & \beta H_2 + C \end{bmatrix}, \quad \mathcal{N}_{\alpha,\beta} = \frac{1}{2}\begin{bmatrix} \alpha H_1 - A & -B^T \\ B & \beta H_2 - C \end{bmatrix}, \tag{2}$$

in which H_1, H_2 are $n \times n$ and $m \times m$ symmetric positive definite matrices, respectively. Similar to the shift-splitting method studied in [23], there exists a unique splitting $\mathcal{A} = \mathcal{M}_{\alpha,\beta} - \mathcal{N}_{\alpha,\beta}$ with $\mathcal{M}_{\alpha,\beta}$ being nonsingular. This special splitting leads to the following new iterative method in fixed point form

$$\mathbf{u}^{k+1} = \mathcal{T}_{\alpha,\beta}\mathbf{u}^k + \mathbf{c} \tag{3}$$

for solving the saddle linear system (1), where $\mathcal{T}_{\alpha,\beta}$ is the iteration matrix given by

$$\mathcal{T}_{\alpha,\beta} = \mathcal{M}_{\alpha,\beta}^{-1}\mathcal{N}_{\alpha,\beta} = \mathcal{I} - \mathcal{M}_{\alpha,\beta}^{-1}\mathcal{A}, \tag{4}$$

and $\mathbf{c} = \mathcal{M}_{\alpha,\beta}^{-1}\mathbf{b}$. From now onward, we use $\mathcal{I}$ (or I) to denote the identity matrix of appropriate size, and we call the stationary iteration (3) as the local generalized shift-splitting (LGSS) iteration method.

Now we discuss the convergence property of the stationary iteration (3). It is well known that the iteration (3) converges for any initial guess $\mathbf{u}^0$ if and only if $\rho(\mathcal{T}_{\alpha,\beta}) < 1$, where $\rho(\mathcal{T}_{\alpha,\beta})$ denotes the spectral radius of the iteration matrix $\mathcal{T}_{\alpha,\beta}$. Since the matrices H_1 and H_2 are symmetric positive definite, let $\widetilde{A} = H_1^{-\frac{1}{2}}AH_1^{-\frac{1}{2}}$, $\widetilde{B} = H_2^{-\frac{1}{2}}BH_1^{-\frac{1}{2}}$, $\widetilde{C} = H_2^{-\frac{1}{2}}CH_2^{-\frac{1}{2}}$. By direct computations we have

$$\mathcal{T}_{\alpha,\beta} = \mathrm{diag}(H_1, H_2)^{-\frac{1}{2}}\widetilde{\mathcal{T}}_{\alpha,\beta}\mathrm{diag}(H_1, H_2)^{\frac{1}{2}}, \tag{5}$$

where

$$\widetilde{\mathcal{T}}_{\alpha,\beta} = \begin{bmatrix} \alpha I + \widetilde{A} & \widetilde{B}^T \\ -\widetilde{B} & \beta I + \widetilde{C} \end{bmatrix}^{-1}\begin{bmatrix} \alpha I - \widetilde{A} & -\widetilde{B}^T \\ \widetilde{B} & \beta I - \widetilde{C} \end{bmatrix}. \tag{6}$$

This means the similarity relationship between $\mathcal{T}_{\alpha,\beta}$ and $\widetilde{\mathcal{T}}_{\alpha,\beta}$. From Theorem 2.1 in [42] we have $\rho(\widetilde{\mathcal{T}}_{\alpha,\beta}) < 1$, then so is $\rho(\mathcal{T}_{\alpha,\beta})$.