Applied Mathematics and Computation 302 (2017) 68–79

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Bridged graphs, circuits and Fibonacci numbers

Marx Stampfli

Bern University of Applied Science, Engineering and Information Technology, PO box Biel 2501, Switzerland

ARTICLE INFO

Keywords:
Two-terminal graphs
Resistor circuits
Bridged networks
Combinatoric counting

ABSTRACT

Series-parallel two-terminal graphs and corresponding unit resistor circuits are well explored. Here we expand the ideas to exclusive-bridged graphs and unit resistor circuits. We proof that both types of circuits have rational resistances whereof numerators and denominators of reduced fractions are smaller or equal to the Fibonacci number F_{n+1}. Series-parallel circuits satisfy another inequality. The sum of their numerator and denominator is smaller or equal to F_{n+2}. This is not true for exclusive-bridged circuits. The consequence is that combinations of these circuits or double-bridged circuits do not satisfy these inequalities. In a second part, counting series-parallel graphs and circuits is expanded to exclusive bridged graphs and circuits. This leads to new terms in two OEIS sequences.

1. Introduction

It is known that series-parallel circuits with n edges have rational resistances whereof numerators and denominators of the reduced fraction are smaller or equal to the Fibonacci number F_{n+1}, Khan [4,5]. Here we proof a slightly stronger theorem. Additionally the sum of numerator and denominator is smaller or equal to F_{n+2}. This fact is a necessary condition. We will see this analyzing the exclusive bridged circuits, which allow the existence of a sometimes called Wheatstone bridge. We proof that exclusive bridged circuits satisfy the first but not the second inequality. The consequences are significant. Exclusive bridged circuits combined with series-parallel circuits do not satisfy these inequalities in general. Even less do the double bridged circuits. In a second part of this paper we count the exclusive bridged graphs and circuits. To count the graphs we need to identify topological identical graphs by determining the symmetry group of exclusive bridged graphs. A still open question is to find a generator function similar to the existing one for series-parallel graphs. In the case of circuits, no generator function is published, not even for the number of series-parallel circuits. Our results allow us to correct an existing and to find further values of sequences in the OEIS Encyclopedia of integer sequences.

2. Definitions

See Harrison [1, p.33–41] and Khan [4,5].

- A **two-terminal graph** is a non separable planar graph with two distinguished different nodes s and t called source and sink, respectively.
- A **two-terminal series-parallel graph (SP graph)** is a graph recursively defined by series and parallel compositions of smaller SP graphs, beginning with the one-edged graph.

E-mail addresses: marx.stampfli@bfh.ch, stampfli@mathematics.ch

http://dx.doi.org/10.1016/j.amc.2016.12.030

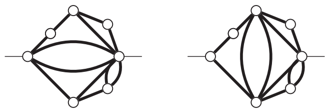

Fig. 1. SP graph (left and EB graph (right).

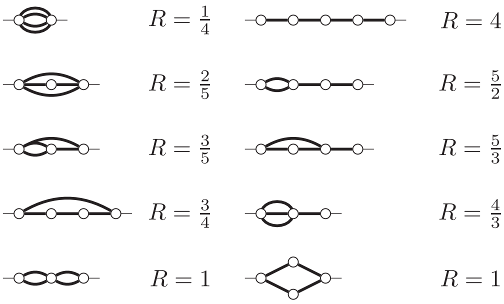

Fig. 2. Four-edged SP graphs ordered in dual pairs and corresponding resistances.

- A **two-terminal exclusive bridged graph (EB graph)** is the graph consisting of five SP graphs forming four peripheral arms and a central arm of a bridged two-terminal graph with s and t not incident to the central arm.
- The **dual graph** of a plane graph is a graph that has a node for each face of the graph and an edge whenever two faces of the graph are separated by an edge.
- A pair of two-terminal graphs are called **duals** if the graphs obtained by adding edges between their sources and sinks are dual graphs.
- The **circuit** corresponding to a two-terminal graph is the circuit that results from replacing every edge of the graph by a resistor of unit resistance.

Examples of a SP and a EB graph are shown in Fig. 1, duality for SP graphs in 2.

Corresponding circuits of a planar graph and its dual have inverse (reciprocal) resistances, since duality in graph theory corresponds to duality in electrical circuits.

3. Graphs and circuits

In the first part we concentrate on circuits, i.e. graphs where the edges are replaced by unit resistors. In physical units the edges of the graph have a resistance of 1Ω. We proof a known fact about SP circuits in a slightly more general form. Especially we need some parts of Theorem 1 to proof Theorem 2.

3.1. SP circuits

Theorem 1. *Let R be the resistance of a circuit corresponding to a SP graph with n edges and F_n the Fibonacci sequence. Then*

$$\text{If } R = \frac{u}{v},\ \gcd(u, v) = 1,\ \text{ then } u \le F_{n+1}, v \le F_{n+1} \text{ and } u + v \le F_{n+2}$$

Proof by induction

Basis

Let $n = 1$. The only single edge circuit has resistance $R = 1$. So $u = 1 \le F_2 = 1,\ v = 1 \le F_2$ and $u + v = 2 \le F_3 = 2$.

Induction step

Let $R_1 = \frac{p}{q}$ and $R_2 = \frac{r}{s}$ be the resistances of two circuits with m and n edges. We assume that $p, q \le F_{n+1}, p + q \le F_{n+2}$ and $r, s \le F_{m+1}, r + s \le F_{m+2}$ We have to show that the serial and the parallel combinations $R_{par} = R_1 + R_2$ and $R_{ser} = \frac{R_1 R_2}{R_1 + R_2}$ written as reduced fractions have numerators u and denominators v with $u, v \le F_{n+m+1},\ u + v \le F_{n+m+2}$.

The resistances of series and parallel combinations are

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}, \quad \frac{1}{\frac{q}{p} + \frac{s}{r}} = \frac{pr}{ps + qr}$$

To include the inequalities of the sums for the maximal numerators and denominators we introduce the variables x and y and write $p = F_{n+1} - x, q = F_n + x$ with $0 \le x \le F_{n-1}$ and $r = F_{m+1} - y, s = F_m + y$ with $0 \le y \le F_{m-1}$. Evidently $qs \le F_{n+1}F_{m+1} \le F_{n+m+1}$ and $pr \le F_{n+m+1}$. There is more to do to find

$$\begin{aligned} ps + qr &= (F_{m+1} - y)(F_n + x) + (F_m + y)(F_{n+1} - x) \\ &= F_{m+1}F_n + F_m F_{n+1} + F_{m+1}x - F_m x - F_n y + F_{n+1} y - 2xy \end{aligned}$$