Applied Mathematics and Computation 000 (2016) 1–13

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the convergence of a high-accuracy conservative scheme for the Zakharov equations

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ARTICLE INFO

Keywords:
High-accuracy conservative scheme
Zakharov equations
Conservative properties
Error estimate
Convergence

ABSTRACT

In this paper, a high-accuracy conservative difference scheme is presented to solve the initial-boundary value problem of the Zakharov equations, which preserves the original conservative properties. The proposed scheme is based on finite difference method. The scheme is second-order accuracy in time and fourth-order accuracy in space. A detailed numerical analysis of the scheme is presented including a convergence analysis result. Numerical examples are given to confirm the proposed scheme is efficient, reliable and of high accuracy.

1. Introduction

In this paper, we focus on a high order compact numerical solver for the Zakharov equations

$$iE_t + Exx - NE = 0, \quad (x,t) \in (a,b) \times (0,T], \tag{1.1}$$

$$N_{tt} - Nxx - (|E|^2)_{xx} = 0, \quad (x,t) \in (a,b) \times (0,T], \tag{1.2}$$

with the initial conditions

$$E(x,0) = E_0(x), \quad N(x,0) = N_0(x), \quad N_t(x,0) = N_1(x), \tag{1.3}$$

and the boundary conditions

$$E(a,t) = E(b,t) = 0, \quad N(a,t) = N(b,t) = 0, \tag{1.4}$$

where $E_0(x)$, $N_0(x)$, and $N_1(x)$ are known smooth functions.

The initial-boundary value problem (1.1)–(1.4) is known to possess the following conservative laws[8]:

$$Q_1(t) = ||E||^2 = Q_1(0), \tag{1.5}$$

$$Q_2(t) = ||E_x||^2 + \frac{1}{2}(||v||^2 + ||N||^2)) + (|E|^2, N) = Q_2(0), \tag{1.6}$$

where $v = -f_x$ and $N_t = f_{xx}$.

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http://dx.doi.org/10.1016/j.amc.2016.10.033
0096-3003/

The Zakharov equations were formerly introduced by Zakharov [36] to describe the propagation of Langmuir waves in an unmagnetized ionized plasma, where the complex function $E(x, t)$ is the slowly varying envelope of the highly oscillatory electric field, the real function $N(x, t)$ is the deviation of the ion density from its equilibrium value. Later, it has become commonly accepted that the Zakharov system is a general model to govern interaction of dispersive and non-dispersive waves. Nowadays, it has been applied to various physical problems, such as the theory of molecular and hydrodynamics.

The theoretical results on existence, uniqueness and regularity of the solution to (1.1)–(1.2) were investigated in [7,10,28]. Various numerical techniques particularly including finite difference method, time-splitting spectral methods, local discontinuous Galerkin method and the Legendre spectral and pseudospectral method have been used for the solution of the Zakharov equations. In this respect, we refer the reader to [2,3,8,9,15,16,19–21,34], and references therein. For wide and interesting topics covered, we should also recall the numerical study done on the equations in [4–6]; especially in [4], an efficient Jacobi pseudospectral method was developed for nonlinear complex generalized Zakharov system.

Recently, there has been growing interest in high-accuracy methods to solve the partial differential equations[1,11–14,18,22–24,26,29,32,33,35], where fourth-order finite difference approximation solutions for the two-dimensional modified anomalous fractional sub-diffusion equation with a nonlinear source term, the coupled nonlinear Schrödinger system, a N-carrier system, the Klein–Gordon equation, the Sine–Gordon equation, the one-dimensional heat and advection–diffusion equations, 2D Rayleigh–Stokes problem, GRLW equation, the KGS equation, the Schrödinger equation and the KGZ equation were shown, respectively. These numerical methods may give us many enlightenments to design a new numerical scheme for the Zakharov equations. In this paper, we propose a high-accuracy compact conservative finite difference scheme to solve the Zakharov system (1.1)–(1.2), whose theoretic accuracy is $O(\tau^2 + h^4)$. The coefficient matrices of the present scheme is symmetric and tridiagonal, so Thomas algorithm can be employed to solve them effectively. Numerical examples are given to confirm the present scheme is of high accuracy and efficient.

The rest of the paper is as follows: In Section 2, we present a high-accuracy compact conservative scheme for the (1+1)-dimensional Zakharov equations. In Section 3, the error estimates and simulation of conservative properties are given. The fourth-order convergence and stability of the scheme are proved in Section 4. In Section 5, numerical experiments are reported to test the theoretical results.

2. High-accuracy compact conservative scheme

Let $h = \frac{b-a}{J}$ and $\tau = \frac{T}{N}$ are the uniform step size in the spatial and temporal direction, respectively, where J and N are two positive integers. Define $\Omega_h = \{x_j = a + jh | 1 \le j \le J-1\}$, $\Omega_\tau = \{t_n = n\tau | 1 \le n \le N-1\}$, $\Omega'_\tau = \{t_n = n\tau | 0 \le n \le N-1\}$, $\bar\Omega_h = \{x_j = a + jh | 0 \le j \le J\}$ and $\bar\Omega_\tau = \{t_n = n\tau | 0 \le n \le N\}$. Denote $E_j^n = E(x_j, t_n)$, $N_j^n = N(x_j, t_n)$, $f_j^n = f(x_j, t_n)$, $U_j^n \approx E(x_j, t_n)$, $V_j^n \approx N(x_j, t_n)$, and $F_j^n \approx f(x_j, t_n)$. Suppose $v = \{v_j^n; j = 0, 1, 2, \ldots, J, n = 0, 1, 2, \ldots, N\}$ be a discrete grid function on $\bar\Omega_h \times \bar\Omega_\tau$. Introduce the following notations:

$$\delta_t v_j^n = \frac{v_j^{n+1} - v_j^n}{\tau}, \quad \delta_{\bar t} v_j^n = \frac{v_j^n - v_j^{n-1}}{\tau}, \quad \delta_{\hat t} v_j^n = \frac{v_j^{n+1} - v_j^{n-1}}{2\tau}, \quad \delta_x v_j^n = \begin{cases} \dfrac{v_{j+1}^n - v_j^n}{h}, & 0 \le j \le J-1, \\ 0, & j = J. \end{cases},$$

$$\delta_{\bar x} v_j^n = \begin{cases} \dfrac{v_j^n - v_{j-1}^n}{h}, & 1 \le j \le J, \\ 0, & j = 0 \end{cases}, \quad \delta_x \delta_{\bar x} v_j^n = \begin{cases} \dfrac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2}, & 1 \le j \le J-1, \\ 0, & j = 0, \\ 0, & j = J. \end{cases}.$$

Let $Z_h^0 = \{\boldsymbol{v} | \boldsymbol{v} = (v_0^n, v_1^n, \ldots, v_J^n), v_0^n = v_J^n = 0\}$. For $\forall \boldsymbol{v}, \boldsymbol{u} \in Z_h^0$, we define the discrete inner products and norms on Z_h^0 via:

$$(\boldsymbol{v}, \boldsymbol{u}) = h \sum_{j=1}^{J-1} v_j^n \overline{u_j^n}, \quad (\delta_x \boldsymbol{v}, \delta_x \boldsymbol{u})_l = h \sum_{j=0}^{J-1} \delta_x v_j^n \delta_x \overline{u_j^n}, \quad ||\boldsymbol{v}||^2 = (\boldsymbol{v}, \boldsymbol{v}),$$
$$||\delta_x \boldsymbol{v}|| = \sqrt{(\delta_x \boldsymbol{v}, \delta_x \boldsymbol{v})_l}, \quad ||\boldsymbol{v}||_\infty = \max_{1 \le j \le J-1} |v_j^n|.$$

We also define the following average operator $\mathscr{A}_h$:

$$\mathscr{A}_h v_j^n = \begin{cases} \dfrac{1}{12}(v_{j+1}^n + 10 v_j^n + v_{j-1}^n), & 1 \le j \le J-1, \\ v_0^n, & j = 0, \\ v_J^n, & j = J. \end{cases}$$

In the rest of this paper, C denotes a general positive constant which may have different values in different occurrences.

For the 2-order derivatives E_{xx} and N_{xx}, we have the following formulas [30]:

$$E_{xx}(x_j) = \mathscr{A}_h^{-1} \delta_x \delta_{\bar x} E(x_j) + O(h^4), \quad N_{xx}(x_j) = \mathscr{A}_h^{-1} \delta_x \delta_{\bar x} N(x_j) + O(h^4), \quad (j \ne 0, J).$$