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Polynomial approximation and quadrature on geographic rectangles☆

M. Gentile, A. Sommariva, M. Vianello*

Department of Mathematics, University of Padova, via Trieste 63, Padova 35121, Italy

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ABSTRACT

Using some recent results on *subperiodic* trigonometric interpolation and quadrature, and the theory of admissible meshes for multivariate polynomial approximation, we study product Gaussian quadrature, hyperinterpolation and interpolation on some regions of $\mathbb{S}^d$, $d \geq 2$. Such regions include caps, zones, slices and more generally spherical rectangles defined on $\mathbb{S}^2$ by longitude and (co)latitude (geographic rectangles). We provide the corresponding Matlab codes and discuss several numerical examples on $\mathbb{S}^2$.

1. Introduction

In this work we study new rules for numerical cubature and define new algorithms to determine good point sets for interpolation on some regions of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ with $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|x\|_2 = 1\}$, being $\|\cdot\|_2$ the euclidean norm in $\mathbb{R}^{d+1}$.

Many cubature and interpolation point sets are known on the whole sphere. Well-known sets are the so-called *spherical L-designs*, introduced by Delsarte et al. [12], that are cubature rules with a fixed algebraic degree of precision and equal weights. Low-cardinality spherical designs (in particular close to the minimal ones) are the most interesting from the computational point of view, see e.g., [38] and references therein. Reimer in [30] and Sloan and Womersley in [32,33], studied the so called *extremal points*, determining good points for interpolation and quadrature; for a survey on this topic, see [22].

Later, Hesse and Womersley in [23] studied numerical integration over caps in $\mathbb{S}^d$ giving regularity results and a lower bound on the cardinality of rules with positive nodes and a certain degree of exactness n. Moreover, they provided rules that have $O(n^d)$ points and degree of exactness n. In particular, exploiting symmetry they presented a rule for caps of $\mathbb{S}^2$ that has $n^2/2 + \mathcal{O}(n)$ points. Using a different approach, Mhaskar showed in [27], under some mild requirements, the existence of certain cubature rules having scattered data as nodes, on domains such as spherical caps and spherical collars. In [28], he generalized these results to more general compact sets of the sphere.

In [2] Beckmann et al. studied integration over spherical triangles providing numerical quadrature rules via certain reproducing kernels techniques.

In this paper, we study *cubature rules* of product Gaussian type on regions of $\mathbb{S}^d$ that we will call “geographic rectangles”, with caps and collars (also called zones) as special cases on $\mathbb{S}^2$. In particular we will determine cubature rules that are exact on all algebraic polynomials of total degree not greater than n, by using “subperiodic” trigonometric Gaussian rules, that are rules with $n+1$ angular nodes, exact on trigonometric polynomials of degree not greater than n on subintervals of

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* Corresponding author.
E-mail addresses: gentimar@hotmail.it (M. Gentile), alvise@math.unipd.it (A. Sommariva), marcov@math.unipd.it (M. Vianello).

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the period, $[\alpha, \beta] \subseteq [0, 2\pi]$ (see [8–11]). We show the quality of the cubature rules by numerical tests on some examples with integrands on $\mathbb{S}^2$ and $\mathbb{S}^4$.

Then, we study function approximation on such regions of the sphere. The availability of algebraic cubature formulas with positive weights, gives the possibility of constructing total-degree *hyperinterpolation* polynomials, that are ultimately truncated and discretized orthogonal polynomial expansions. Such a technique was introduced by Sloan in the seminal paper [31], and then developed in various contexts, as a valid alternative to polynomial interpolation; see, e.g., [20,21,34] and references therein. Orthogonal polynomials on the relevant regions, which are a key ingredient of hyperinterpolation, are here computed by numerical linear algebra methods (consecutive *QR* factorizations of weighted Vandermonde matrices).

Such a connection with *hyperinterpolation on regions* of the sphere is one of the main motivations to construct cubature formulas that are *exact* on total-degree polynomials. Indeed, concerning pure cubature, some preliminary numerical experiments seem to suggest that near-exactness (say, with an error not far from machine precision) can be obtained also by the product Gauss–Legendre quadrature in the angular variables, and even that a subsampling phenomenon can arise (provided that the angular intervals are sufficiently small). Such numerical observations, that go beyond the scope of the present paper, deserve in any case further deepening, as well as a comprehensive future study from both the computational and the theoretical sides.

On the other hand, the recently developed theory of subperiodic trigonometric interpolation, cf. [5], allows us to construct *Weakly Admissible Meshes* (shortened as WAMs) on geographic rectangles. The theory of WAMs, which are essentially special sequences of finite norming sets for polynomial spaces, has been introduced by Calvi and Levenberg in the seminal paper [7], and has been developed by various researchers in the last years; cf., e.g., [4,24,29]. In the present context, product-type WAMs on geographic rectangles are straightforward to compute for any degree, and can be used directly for least-squares approximation of continuous functions (near-optimal in the uniform norm). Furthermore, by the algorithms described in [35], we extract from such WAMs the so called *Approximate Fekete Points* and *Discrete Leja Points*. Both these point sets are good for polynomial interpolation, since they are asymptotically distributed as the Fekete points of the region and enjoy a slowly increasing Lebesgue constant; cf., e.g., [3].

All the Matlab codes used for the numerical experiments are available at the web site [6].

2. Some basic definitions and results

As preliminaries, it is important to give a quick glance to some well-known facts that will be important in the next sections. We will denote by $\mathbb{P}_n(\Omega)$ the space of he restrictions to Ω of the algebraic polynomials of total degree at most n in $\mathbb{R}^{d+1}$. A standard parameterization of the sphere $\mathbb{S}^d$ is provided by *generalized spherical coordinates* as

$$x_k = \begin{cases} \cos(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j), \; k = 1, \\ \sin(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j), \; k = 2, \\ \cos(\theta_{d+2-k}) \cdot \prod_{j=1}^{d+1-k} \sin(\theta_j), \; k = 3, \dots, d+1 \end{cases} \tag{1}$$

with the notation $\prod_{j=1}^{0} \sin(\theta_j) \equiv 1$. A classical choice in the range of the angles is $\theta_d \in [0, 2\pi)$ and $\theta_k \in [0, \pi]$ for $k = 1, \dots, d-1$.

We point out that depending on the authors this parameterization may change. Independently of the choice of the range of the angles, the surface measure is expressed as

$$d\mu(\mathbf{x}) = \prod_{k=1}^{d-1} \sin^{d-k}(\theta_k) d\theta_k.$$

We will denote by $\xi = \xi(\theta_1, \dots, \theta_d)$ the transformation from generalized spherical coordinates to cartesian coordinates.

In the case $d = 2$, setting $\theta := \theta_1$, $\phi := \theta_2$, we have in particular the usual spherical coordinates transformation $\xi = \xi(\theta, \phi)$ defined by

$$\begin{aligned} x_1 &= \cos(\phi) \cdot \sin(\theta), \\ x_2 &= \sin(\phi) \cdot \sin(\theta), \\ x_3 &= \cos(\theta) \end{aligned} \tag{2}$$

with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, and surface measure $\sin(\theta)$.

The *spherical harmonics* $\mathbb{H}_k(\mathbb{S}^d)$ of (exact) degree k (cf. [1, p.133]) are widely used to determine a basis on the sphere $\mathbb{S}^d$. They are homogenous polynomials of degree k

$$p(x_1, \dots, x_{d+1}) = \sum_{b_1+\dots+b_d=k} a_{b_1,\dots,b_{d+1}} x_1^{b_1} \dots x_{d+1}^{b_{d+1}}$$

such that

$$\Delta p(x_1, \dots, x_{d+1}) = 0$$