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The normalized Laplacian spectrum of quadrilateral graphs and its applications

Deqiong Li, Yaoping Hou*

College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China

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ABSTRACT

The quadrilateral graph $Q(G)$ of G is obtained from G by replacing each edge in G with two parallel paths of lengths 1 and 3. In this paper, we completely describe the normalized Laplacian spectrum on $Q(G)$ for any graph G. As applications, the significant formulae to calculate the multiplicative degree-Kirchhoff index, the Kemeny's constant and the number of spanning trees of $Q(G)$ and the quadrilateral iterative graph $Q_r(G)$ are derived.

1. Introduction

Spectral graph theory tries to derive information about graphs from the graph spectrum [5,6]. There is extensive literature on works related to the spectrum on various matrices such as adjacency, Laplacian and normalized Laplacian matrices. Especially in recent years, the normalized Laplacian, which is consistent with the eigenvalues in spectral geometry and in random processes, has attracted increasing attention from researchers because many results which were only known for regular graphs can be generalized to all graphs.

Let G be a simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. An edge connecting two vertices $i, j \in V(G)$ is denoted by ij. If $ij \in E(G)$, we say i is a neighbor of j and write as $i \sim j$ or we say i and j are adjacent. If e is an edge with end-vertices i and j, then we say that e and i or e and j are incident. The degree of a vertex i is denoted by d_i. Let A be the adjacency matrix of G, and D be the diagonal matrix of vertex degree of G. The matrix $L = D - A$ is called the Laplacian matrix.

The *random walk* is defined as the Markov chain $X_n (n \geq 0)$, that from its current vertex i jumps arbitrarily to its neighboring vertex j with probability $p_{ij} = \frac{1}{d_i}$. We denote by $M = (p_{ij})$ the *transition probabilities matrix* for random walks on G. So

$$p_{ij} = \begin{cases} \frac{1}{d_i} & \text{if } \ i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $M = D^{-1}A$ is a stochastic matrix.

The *normalized Laplacian* is defined to be

$$\mathfrak{L} = I - D^{\frac{1}{2}} M D^{-\frac{1}{2}},$$

* Corresponding author. Fax: +86 731 88827267.
E-mail addresses: yphou@hunnu.edu.cn, yphou9898@gmail.com (Y. Hou).

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where I is the identity matrix with the same order as M. Let δ_{ij} be the Kronecker delta. From the definition of $\mathfrak{L}$, we have the following relationship easily:

$$\mathfrak{L}(i,j) = \delta_{ij} - \frac{A(i,j)}{\sqrt{d_i d_j}},$$

where $\mathfrak{L}(i,j)$ and $A(i,j)$ denote the (i,j)-entry of $\mathfrak{L}$ and A respectively. Since $\mathfrak{L}$ is Hermitian and similar to $I - M = D^{-1}L$, the eigenvalues of $\mathfrak{L}$ are non-negative. We label the eigenvalues of $\mathfrak{L}$ so that $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$, where n is the number of vertices of G. The spectrum on the normalized Laplacian matrix $\mathfrak{L}$ of the graph G is defined as $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, which also is called the normalized Laplacian spectrum of G. The normalized Laplacian spectrum of a graph offers us the relate structural information about the graph [5].

Lemma 1. *[5] Let G be a connected graph with n vertices, and $\mathfrak{L}$ be the normalized Laplacian of G. The normalized Laplacian spectrum of G is $\sigma = \{0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n\}$. We have*

(i) *For all $i \le n$, we have $\frac{n}{n-1} \le \lambda_i \le 2$ with $\lambda_n = 2$ if and only if G is bipartite;*
(ii) *G is bipartite if and only if λ_i is an eigenvalue of $\mathfrak{L}$, then the value $2 - \lambda_i$ is also an eigenvalue of $\mathfrak{L}$ and $m_{\mathfrak{L}}(\lambda_i) = m_{\mathfrak{L}}(2 - \lambda_i)$, where $m_{\mathfrak{L}}(\lambda_i)$ denotes the multiplicity of the eigenvalue λ_i of $\mathfrak{L}$.*

Different from the standard distance between two vertices, which is defined as the length of a shortest path that connects these two vertices, Klein and Randić [14] introduced a new distance function named resistance distance. The *resistance distance* between two vertices i and j of a graph G, denoted by r_{ij}, is the electrical resistance between i and j when placing a unit resistor on every edge and a battery is attached at i and j. Similar to the standard distance, the resistance distance is also intrinsic to the graph, not only with some fine purely mathematical properties, but also with a substantial potential for chemical applications. Recently, in [4] a new index named the *multiplicative degree-Kirchhoff index* is introduced. It is defined as

$$\boldsymbol{Kf}'(G) = \sum_{i<j} d_i d_j r_{ij}.$$

The multiplicative degree-Kirchhoff index has a very close association with the normalized Laplacian spectrum (Lemma 2(i)). Many results about the normalized Laplacian spectrum and the multiplicative degree-Kirchhoff index of some graphs have been obtained [2,9–13,15,18,19].

The *Kemeny's constant* $K(G)$ of G is the expected number of steps required for the transition from a starting vertex i to a destination vertex, which is chosen randomly according to a stationary distribution of unbiased random walks on G. The Kemeny's constant gives an interesting quantity for finite ergodic Markov chains, which is independent of the initial state of the Markov chain [8].

In terms of the spectrum on the normalized Laplacian of G, the special calculation formulae for the multiplicative degree-Kirchhoff index, the Kemeny's constant and the number of spanning trees of graph G can be expressed as follows.

Lemma 2. *Let G be a connected graph with n vertices and m edges, $\sigma = \{0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n\}$ is the spectrum on the normalized Laplacian $\mathfrak{L}$ of G. then*

(i) *[4] The multiplicative degree-Kirchhoff index of G is*

$$\boldsymbol{Kf}'(G) = 2m \sum_{i=2}^{n} \frac{1}{\lambda_i}.$$

(ii) *[1] The Kemeny's constant of G is*

$$K(G) = \sum_{i=2}^{n} \frac{1}{\lambda_i}.$$

(iii) *[5] The number $\kappa(G)$ of spanning trees of G is*

$$\kappa(G) = \frac{1}{2m} \prod_{i=1}^{n} d_i \prod_{k=2}^{n} \lambda_k.$$

Apparently, from Lemma 2 (i) and (ii) the straightforward relation between the multiplicative degree-Kirchhoff index and the Kemeny's constant is

$$\boldsymbol{Kf}'(G) = 2mK(G). \tag{1}$$

Let G be a simple and connected graph with n vertices and m edges. Replacing each edge of G with two parallel paths of lengths 1 and 3 results in a new graph $Q(G)$, which is called the *quadrilateral graph* of the graph G. Another view of constructing $Q(G)$ is follows: For each edge ij of G, we add two new vertices i', j' and three new edges ii', $i'j'$, $j'j$. Fig. 1 gives an example of the quadrilateral graph of the cycle C_4.