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# Blow-up rates of large solutions for infinity Laplace equations☆

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ABSTRACT

In this paper, by constructing suitable comparison functions, we mainly give the boundary behavior of solutions to boundary blow-up elliptic problems $\triangle_\infty u = b(x)f(u),\ x \in \Omega,\ u|_{\partial\Omega} = +\infty$, where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$, the operator $\triangle_\infty$ is the $\infty$-Laplacian, $b \in C^\alpha(\bar{\Omega})$ which is positive in $\Omega$ and may be vanishing on the boundary and rapidly varying near the boundary and the nonlinear term $f$ is a $\Gamma$-varying function at infinity, whose variation at infinity is not regular.



## 1. Introduction and the main results

This paper is devoted to the study of asymptotic behavior of solutions to the following boundary blow-up elliptic problem

$$\triangle_\infty u = b(x)f(u),\ \ u > 0,\ \ x \in \Omega,\ \ \ u|_{\partial\Omega} = \infty, \tag{1.1}$$

where the operator $\triangle_\infty$ is the $\infty$-Laplacian, a highly degenerate elliptic operator given by

$$\triangle_\infty u := \langle D^2uDu, Du\rangle = \sum_{i,j=1}^{N} D_iuD_{ij}uD_ju,$$

$b$ satisfies

**($\mathbf{b_1}$)** $b \in C(\bar{\Omega})$ is positive in $\Omega$,

and $f$ satisfies

**($\mathbf{f_1}$)** $f \in C[0, \infty) \cap C^1(0, \infty)$, $f(0) = 0$, $f(s) > 0, s > 0$, $f(s)/s$ is increasing on $(0, \infty)$.

By a solution to the problem (1.1), we mean a nonnegative function $u \in C(\Omega)$ that satisfies the equation in the viscosity sense (see Section 2 for definition) and the boundary condition with $u(x) \to \infty$ as the distance function $d(x) := dist(x, \Omega)$ $\to 0$. Such a solution is called a large solution, an explosive solution or a boundary blow-up solution.

The $\infty$-Laplacian was first introduced in the work of Aronsson [1] in connection with the geometric problem of finding the so-called absolutely minimizing functions in $\Omega$. However, this operator is quasilinear and highly degenerate and in

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general does not have smooth solutions. Therefore solutions are understood in the viscosity sense, a concept introduced by Crandall and Lions [2] and Crandall et al. [3]. By using the viscosity solutions, Jensen [4] proved that $u \in C(\bar{\Omega})$ is an absolute minimizing Lipschitz extension of $g \in Lip(\partial\Omega)$ if and only if $\triangle_\infty u = 0$, in the viscosity sense, in $\Omega$ and $u = g$ on $\partial\Omega$. Since then, the infinity Laplace equation has been extensively studied, see, for instance, [5–12] and the references therein.

The investigation of boundary blow-up problems for elliptic equations has a long history. Early studies mainly focused on problems involving the classical Laplace operator $\Delta$, i.e.

$$\Delta u = b(x)f(u), \ u > 0, \ x \in \Omega, \ u|_{\partial\Omega} = \infty. \tag{1.2}$$

The problem (1.2) was considered for the first time by Bieberbach [13] with $N = 2, b(x) = 1$ and $f(u) = e^u$, the author showed that there exists a unique solution such that $u(x) - log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$. Problems of this type arise in Riemannian geometry, more precisely, if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has constant Gaussian curvature $-b^2$, then $\Delta u = b^2e^{2u}$. It was finally shown by Lazer and McKenna in [14] that the solution is unique with no further restriction.

For $b(x) \equiv 1$, Keller-Osserman ([15,16]) supplied a necessary and sufficient condition

$$\int_a^\infty \frac{d\nu}{\sqrt{2F(s)}} < \infty, \quad \forall a > 0, \quad F(s) = \int_0^s f(\nu)d\nu, \tag{1.3}$$

for the existence of solutions to problem (1.2).

Loewner and Nirenberg [17] showed that if $f(u) = u^{p_0}$ with $p_0 = (N+2)/(N-2)$, $N > 2$, then problem (1.2) has a unique positive solution $u$ which satisfies

$$\lim_{d(x)\to 0} u(x)(d(x))^{(N-2)/2} = \Big(N(N-2)/4\Big)^{(N-2)/4}.$$

Bandle and Marcus [18] established the following results: if $f$ satisfies $(f_1)$ and the condition that

$(H_0)$ there exist $\theta > 0$ and $S_0 \geq 1$ such that $f(\xi s) \leq \xi^{1+\theta} f(s)$ for all $\xi \in (0, 1)$ and $s \geq S_0/\xi$, then for any solution $u$ of problem (1.2)

$$\frac{u(x)}{\phi(d(x))} \to 1 \quad as \quad d(x) \to 0, \tag{1.4}$$

where $\phi$ satisfies

$$\int_{\phi(t)}^\infty \frac{ds}{\sqrt{2F(s)}} = t, \ \forall t > 0. \tag{1.5}$$

If $f$ further satisfies

$(H_1)$ $f(s)/s$ is increasing in $(0, \infty)$, then problem (1.2) has a unique solution.

Lazer-McKenna [19] showed that if $f$ satisfies $(f_1)$ and

$(H_2)$ there exists $S_0 \geq 0$ such that $f'$ is non-decreasing on $[0, \infty)$ and

$$\lim_{s\to\infty} \frac{f'(s)}{\sqrt{F(s)}} = \infty,$$

then for any solution $u$ of problem (1.2)

$$u(x) - \phi(d(x)) \to 0 \ as \ d(x) \to 0.$$

Now we introduce a class of functions.

Let $\Lambda$ denote the set of all positive non-decreasing functions $k \in C^1(0, \nu)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = C_k, \quad \text{where} \quad K(t) = \int_0^t k(s)ds. \tag{1.6}$$

We note that for each $k \in \Lambda$,

$$\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0 \text{ and } C_k \in [0, 1].$$

The set $\Lambda$ was first introduced by Cîrstea and Rădulescu in [20–24].

To present our main results, we briefly recall some notions from Karamata's theory.

A positive measurable function $f$ defined on $[a, \infty)$, for some $a > 0$, is called **regularly varying at infinity** with index $\rho$, written $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{1.7}$$

In particular, when $\rho = 0$, $f$ is called **slowly varying at infinity**.