Applied Mathematics and Computation 298 (2017) 141–152

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Nine limit cycles around a singular point by perturbing a cubic Hamiltonian system with a nilpotent center

Junmin Yang[a], Pei Yu[b,*]

[a] College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, PR China
[b] Department of Applied Mathematics, Western University, London, Ontario N6A 5B7, Canada

ARTICLE INFO

Keywords:
Near-Hamiltonian system
Nilpotent center
Hopf bifurcation
Limit cycle
Normal form
Generalized Lyapunov constant

ABSTRACT

In this paper, we study bifurcation of limit cycles in planar cubic near-Hamiltonian systems with a nilpotent center. We use normal form theory to compute the generalized Lyapunov constants and show that there exist at least 9 limit cycles around the nilpotent center. This is a new lower bound on the number of limit cycles in planar cubic near-Hamiltonian systems with a nilpotent center.

1. Introduction

It is well known that dynamical systems can exhibit self-sustained oscillations, called limit cycles, which may appear in almost all fields of science and engineering such as physics, mechanics, electronics, ecology, economy, biology, finance etc. Developing theory and methodology for solving limit cycle problems is not only theoretically significant, but also practically important. The phenomenon of limit cycles was first discovered and introduced by Poincaré who developed the breakthrough qualitative method, the Poincaré Map [1], to determine the existence of limit cycles, which is still the most basic tool for studying stability and bifurcation of periodic orbits. Later, many quantitative methodologies were developed to approximate the solution of limit cycles, in particular bifurcating from Hopf critical points, for example, see [2] and reference therein. Recently, with the aid of computer algebra systems such as Mathematica, Maple, symbolic algorithms and programs have been developed to overcome the computational complexity in the analysis of bifurcation of limit cycles, for example, see [3] in which many practical problems are presented and solved by using limit cycles theory and normal form theory. Very recently, bifurcation of multiple limit cycles in disease models has attracted attention of researchers in this field, since such bifurcation can cause complex biological behavior like bistable states which may involve equilibria and periodic motions. For example, in a simple 2-dimensional in-host model of HIV, developed in [4–6], besides studying the interesting phenomenon–viral blips, bifurcation of two limit cycles from a Hopf critical point has been found in [6]. These two limit cycles enclosing a stable equilibrium with the outer cycle stable indeed show that depending upon different initial conditions, the system trajectories can either converge to the disease-free equilibrium or to a stable periodic motion of disease.

The development of limit cycles theory is closely related to the well-known Hilbert's 16th problem, one of the 23 mathematical problems proposed by Hilbert at the Second International Congress of Mathematics in 1900 [7]. A modern version of this problem was later formulated by Smale, chosen as one of his 18 most challenging mathematical problems for the 21st century [8]. To be more specific, consider the following planar differential system:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{1.1}$$

* Corresponding author.
E-mail address: pyu@uwo.ca (P. Yu).

http://dx.doi.org/10.1016/j.amc.2016.11.021
0096-3003/

where $P_n(x, y)$ and $Q_n(x, y)$ represent nth-degree polynomials in x and y. The second part of Hilbert's 16th problem is to find an upper bound on the number of limit cycles that system (1.1) can have. This upper bound, called Hilbert number, is the function of n only denoted as $H(n)$. In early 1990's, Ilyashenko and Yakovenko [9], and Écalle [10] independently proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, four limit cycles were obtained more than 30 years ago [11,12]. This result was also proved recently for near-integrable quadratic systems [13]. However, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number, and the best result so far is $H(3) \geq 13$ [14,15]. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [16]. It should be pointed out that in real applications, many systems have dimension higher than two [3–5] and Hopf bifurcation leading to limit cycles is a common phenomenon. In such case, the system can be reduced to a 2-dimensional dynamical system by using center manifold theory (e.g., see [3,17]) to study the limit cycles bifurcation.

Later, Arnold [18] posed the weak Hilbert's 16th problem, which is closely related to the so-called near-Hamiltonian system [19]:

$$\dot{x} = H_y(x, y) + \varepsilon\, p_n(x, y), \qquad \dot{y} = -H_x(x, y) + \varepsilon\, q_n(x, y), \tag{1.2}$$

where $H(x, y)$, $p_n(x, y)$ and $q_n(x, y)$ are all polynomial functions in x and y, and $0 < \varepsilon \ll 1$ represents a small perturbation. Then, the problem on bifurcation of limit cycles for such a system can be transferred to studying the zeros of the Abelian function or the (first-order) Melnikov function, given in the form of

$$M(h, \delta) = \oint_{H(x,y)=h} q_n(x, y)\, dx - p_n(x, y)\, dy, \tag{1.3}$$

where $H(x, y) = h$ for $h \in (h_1, h_2)$ defines a closed orbit, and δ represents the parameters (or coefficients) involved in the polynomial functions $p_n(x, y)$ and $q_n(x, y)$.

If the problem is restricted to the vicinity of an isolated fixed point, which is either an elementary center or an elementary focus, then it is equivalent to study generalized Hopf bifurcations. This problem is usually called local bifurcation of limit cycles, and the number of bifurcating small-amplitude limit cycles is denoted by $M(n)$. The best-known result is $M(2) = 3$, which was obtained by Bautin in 1952 [20]. For $n = 3$, a number of results have been obtained. Around an elemental focus, James and Lloyd [21] considered a particular class of cubic systems to obtain 8 limit cycles in 1991, and the systems were reinvestigated couple of years later by Ning et al. [22] to find another solution of 8 limit cycles. Yu and Corless [23] constructed a cubic system and combined symbolic and numerical computations to show 9 limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [24]. Another cubic system was also recently constructed by Lloyd and Pearson [25] to show 9 limit cycles with purely symbolic computation.

On the other hand, around a center, there are also a few results obtained in the past two decades. In 1995, Żołądek [26] first proposed a rational Darboux integral, and claimed the existence of 11 small-amplitude limit cycles around a center. After more than ten years, another two cubic systems are constructed to show 11 limit cycles [27,28]. Recently, based on the system given in [27], 12 small-amplitude limit cycles around a singular point has been proved [29], which is perdhaps the maximal numver which can be obtained for cubic integrable polynomial systems. The system considered in [26] was reinvestigated by Yu and Han [30] using the method of focus value computation, and only 9 limit cycles were obtained. More recently, Tian and Yu proved the Żołądek's example can indeed have only 9 limit cycles [31].

For the local bifurcation problem associated with a singularity of focus, Lyapunov constants are needed to solve the center-focus problem and determine the number and stability of bifurcating limit cycles. There are mainly three approaches which are widely used to compute the Lyapunov constants: the method of normal forms [3,32,33], the method of Poincaré return map or focus value method [34,35], and the method of Lyapunov function [36,37]. Other approaches can be found, for example, in [3]. Since in this paper we apply the method of normal forms to study bifurcation of limit cycles, in the following we briefly describe how this method is used to determine the number of bifurcating limit cycles. Without loss of generality, suppose that system (1.1) has a singularity of focus at the origin, that is, $(x, y) = (0, 0)$ is an equilibrium of system (1.1) and that the Jacobian of the system evaluated at this equilibrium is in the real Jordan canonical form,

$$J = \begin{bmatrix} 0 & \omega_c \\ -\omega_c & 0 \end{bmatrix}.$$

Then, by using the method of normal forms with the aid of computer algebra systems (e.g., see [3,33,38,39]) we can obtain the normal form of the system, given in polar coordinates, as

$$\begin{aligned} \dot{r} &= r\left(\upsilon_0 + \upsilon_1 r^2 + \upsilon_2 r^4 + \cdots + \upsilon_k r^{2k} + \cdots\right), \\ \dot{\theta} &= \omega_c + \tau_0 + \tau_1 r^2 + \tau_2 r^4 + \cdots + \tau_k r^{2k} + \cdots, \end{aligned} \tag{1.4}$$

where r and θ represent the amplitude and phase of motion, respectively. υ_k $(k = 0, 1, 2, \ldots)$ is called the kth-order focus value. υ_0 and τ_0 are obtained from linear analysis. The first equation of (1.4) can be used for studying bifurcation of limit cycles and stability of bifurcating limit cycles, while the second equation can be used to determine the frequency of bifurcating periodic motion. Moreover, the coefficients τ_j's can be used to determine the order (or critical periods) of a center (i.e., when $\upsilon_j = 0$, $j \geq 0$). These focus values are equivalent to Lyapunov constants, L_j, in the sense that

$$\upsilon_j = 0,\ j = 0, 1, \cdots, k-1,\ \upsilon_k \neq 0 \iff L_j = 0,\ j = 0, 1, \ldots, k-1,\ L_k \neq 0,$$