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Dynamical behavior of a stochastic two-species Monod competition chemostat model☆

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ABSTRACT

This paper studies a stochastic two-species Monod competition chemostat model which is subject to environment noises. Such noises are described by independent standard Brownian motions. It proves that the initial value problem of the model has a unique positive global solution. However, unlike the corresponding deterministic model, the stochastic model no longer has positive equilibrium points. The asymptotic behaviors and the steady state distributions are established by using Itô's formula, Lyaponov method and Gronwall inequality. In addition, numerical simulations are given to illustrate the theoretical results.

1. Introduction

The chemostat, a laboratory apparatus used for the continuous culture of microorganisms, plays an important role in mathematical biology and theoretical ecology. It has been extensively investigated by a number of authors, see [1–9] and the references therein. For example, Smith and Waltman considered in [9] the deterministic unscaled version competitive model with a single nutrient and two different microorganisms. The model takes the following form:

$$\begin{cases} S'(t) = (S^0 - S(t))D - \dfrac{m_1 S(t)x(t)}{a_1 + S(t)} - \dfrac{m_2 S(t)y(t)}{a_2 + S(t)}, \\ x'(t) = \dfrac{m_1 S(t)x(t)}{a_1 + S(t)} - Dx(t), \\ y'(t) = \dfrac{m_2 S(t)y(t)}{a_2 + S(t)} - Dy(t), \end{cases} \tag{1.1}$$

where $S(t)$, $x(t)$ and $y(t)$, respectively, stand for the concentrations of the nutrient and two microorganisms at time t, S^0 and D are positive constants, which respectively represent the input concentration of the nutrient and the common washout rate, the functions

$$\frac{m_i S}{a_i + S}, \quad (i = 1, 2)$$

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represent the Monod growth functional response and $m_i(i=1,2)>0$ are called the maximal growth rates, and $a_i(i=1,2)>0$ are the half-saturation constants.

For $i=1$ and 2, we denote

$$\lambda_i=\frac{a_iD}{m_i-D}$$

which are the break-even concentrations of system (1.1) and completely determine the dynamical behaviors of system (1.1). In any case, we find that $E_0(S^0, 0, 0)$ is an equilibrium of (1.1). When $0<\lambda_1<S^0$ or $0<\lambda_2<S^0$, respectively, system (1.1) exists another two equilibriums $E_1=(\lambda_1, S^0-\lambda_1, 0)$ and $E_2=(\lambda_2, 0, S^0-\lambda_2)$. Naturally, the condition $m_i>D$ $(i=1,2)$ should be added. In this case, Smith and Waltman [9] obtained the following results (see Theorem 5.1 or Appendix F in [9]).

Theorem 1.1. *If $\lambda_i>S^0$ $(i=1,2)$, system (1.1) has a globally asymptotically stable washout equilibrium $E_0(S^0, 0, 0)$; If $0<\lambda_1<\lambda_2<S^0$, system (1.1) has a globally asymptotically stable boundary equilibrium $E_1=(\lambda_1, S^0-\lambda_1, 0)$, then the population x is the winner of the competition; If $0<\lambda_2<\lambda_1<S^0$, system (1.1) has another globally asymptotically stable boundary equilibrium $E_2=(\lambda_2, 0, S^0-\lambda_2)$, then the population y is the winner of the competition.*

If $\lambda_1=\lambda_2$, then two microorganisms x and y can coexist, but this is a “knife-edge” effect and cannot be expected to be found in nature [9]. Therefore, the coexistence of two microorganisms x and y is not discussed in this paper.

Clearly, these important and useful works on deterministic chemostat model provide a great insight into the development of environmental microorganism. However, in the real world, the development of microorganism is inevitably perturbed by various types of environment noises. Development from the deterministic models to the stochastic models can give us new insights into the development of environmental microorganism. Indeed, by introducing (stochastic) environmental noise, some scholars have proposed some stochastic epidemic models [10–13], stochastic population models [14–16] and stochastic chemostat models [17–21]. They focus on the effect of the noise on the dynamic behavior of these stochastic models.

We note that there are different possible approaches to include random effects in the model from biological and mathematical perspective, see Beddington and May [22]. In model (1.1), the single nutrient and two microorganisms can be regarded as some particles. When they are put in a chemostat, the Brownian motion will clearly occur. On the other hand, it also has the biological motivation of putting the environmental noise into the density-independent term in model (1.1), see Beddington and May [22] and Horn [23]. In fact, Imhof and Walcher [17] have obtained a stochastic chemostat model by using the discrete Markov chain, proved that $X^{(\Delta t)}(t)$ converges weakly to the solution of the stochastic differential equations as $\Delta t\to 0$, and also given the existence result of strong solution. In addition, they also established some stochastic persistence results for the stochastic model and proved that environment noise may lead to extinction even if the deterministic model predicts persistence.

In this paper, by the view points of Imhof and Walcher [17], we assume that stochastic perturbations are of a white noise type which are directly proportional to $S(t)$, $x(t)$, $y(t)$, influenced on the $S'(t)$, $x'(t)$, $y'(t)$ in the model (1.1). By this way, the model (1.1) will be deduced to the form

$$\begin{cases} \mathrm{d}S(t)=\left[(S^0-S(t))D-\dfrac{m_1S(t)x(t)}{a_1+S(t)}-\dfrac{m_2S(t)y(t)}{a_2+S(t)}\right]\mathrm{d}t+\sigma_1S(t)\mathrm{d}B_1(t),\\ \mathrm{d}x(t)=\left(\dfrac{m_1S(t)x(t)}{a_1+S(t)}-x(t)D\right)\mathrm{d}t+\sigma_2x(t)\mathrm{d}B_2(t),\\ \mathrm{d}y(t)=\left(\dfrac{m_2S(t)y(t)}{a_2+S(t)}-y(t)D\right)\mathrm{d}t+\sigma_3y(t)\mathrm{d}B_3(t), \end{cases} \tag{1.2}$$

where $B_i(t)(i=1,2,3)$ are independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$; $\sigma_i\geq 0(i=1,2,3)$ are constants which reflect the size of the stochastic effects, if the noise intensity $\sigma_i=0(i=1,2,3)$, then model (1.2) turns into model (1.1). The meaning of other parameters are similar to model (1.1).

To investigate the dynamics of stochastic chemostat model (1.2), we need to show at least this model has a unique global positive solution. In the next section, the global existence and positivity of the solutions of system (1.2) will be obtained by constructing an auxiliary function and using Itô's formula and Gronwall's inequality. Thus, our method is different from Imhof and Walcher [17].

Noting that E_0, E_1 and E_2 are not already equilibriums of stochastic system (1.2), it is natural to discuss what will happen to system (1.2) when the conditions of Theorem 1.1 were satisfied. The equilibrium states are now described by a probability distribution. Their computational methods and the asymptotic behaviors of solutions will be established in Section 3. The similar work has also been considered by Mao [24] and Lahrouz and Omari [25]. However, our nonlinear term is different from [24] and [25].

Finally, numerical simulations are carried out to illustrate the theoretical results in Section 4, and the conclusion in ecology will be given in Section 5.

Denote $R^3_+=\{(S,x,y)\in R^3: S>0, x>0, y>0\}$ throughout the paper. Some fundamental knowledge and symbols for the stochastic differential equations can be seen in [26].