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Solving Helmholtz equation at high wave numbers in exterior domains☆

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ABSTRACT

This paper presents effective difference schemes for solving the Helmholtz equation at high wave numbers. Considering the problems in the polar and spherical coordinates, we show that pollution-free difference schemes can be constructed for annulus and hollow sphere domains, but the "pollution effect" cannot be avoided in the neighborhood of the origin of the coordinate. Using the fact that the "pollution effect" appears in a small neighborhood of the origin of the coordinate, we propose a local mesh refinement approach so that a fine grid is applied only for a small region near the origin of the coordinate and a much larger grid size is employed in the remaining domain. The most attractive feature of this approach is that the computational storage can be significantly reduced while keeping almost the same numerical accuracy. Moreover, by applying a two-level local refinement technique, we can further enhance the performance of the proposed scheme. Numerical simulations are reported to verify the effectiveness of the proposed schemes for the Helmholtz problem in exterior domains.

1. Introduction

In this paper, we consider the Helmholtz equation in the exterior domain defined as follows:

$$-\Delta u - k^2 u = 0, \quad \text{in } \mathbb{R}^d \backslash \mathcal{B}_2, \tag{1}$$

$$u|_{\partial \mathcal{B}_2} = g_1, \tag{2}$$

$$\partial_r u - jku = o\left(||x||^{\frac{1-d}{2}}\right), \quad \text{as } ||x|| \to \infty, \tag{3}$$

where k is the wave number, $\mathcal{B}_2$ is a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$, g_1 is a given function, ∂_r denotes the radial derivative and $j^2 = -1$. The solution of (1)–(3) may be reduced to the solution of the following problem (see [5,10,11,17,26,27,37]):

$$-\Delta u - k^2 u = 0, \quad \text{in } \Omega := \mathcal{B}_1 \backslash \mathcal{B}_2, \tag{4}$$

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$$u|_{\partial\mathcal{B}_2} = g_1, \tag{5}$$

$$\left(\partial_r u - jku\right)|_{\partial\mathcal{B}_1} = g_2, \tag{6}$$

where $\mathcal{B}_1 \in \mathbb{R}^d$ $(d = 2, 3)$ is a sufficiently large ball containing $\mathcal{B}_2$ and g_2 is a given function.

It is well-known that solving the Helmholtz equation at high wave numbers numerically is very difficult and challenging, because the solutions are highly oscillatory. Even though the equation is linear, the resulting discrete linear system is indefinite and ill-conditioned (see [1,8,9,19–21,32]). Another difficulty in simulating the numerical solutions at high wave number problems is that the "pollution effect" exists in almost all computational schemes, and consequently the numerical accuracy becomes totally unacceptable for cases with high wave numbers. Many studies were reported on this topic in the past decades, for example, Babuška et al. [2,30] considered the generalized finite element method to minimize the "pollution effect". Another popular technique is based on the $h - p$ finite element method (4)–(6) (see [12,20,21]), in which the "pollution effect" is reduced by increasing the order p of the polynomial basis function or decreasing the mesh size h. For the finite difference methods, many higher order compact schemes were developed [14,25,28,29]. Recently, Chen et al. [10,11] proposed two methods to minimize the numerical dispersion by choosing optimal parameters. Other computational techniques such as the spectral methods were investigated, and the reader is referred to [4,6,13,18,23,26,27,39] and the references therein.

Generally speaking, due to the "pollution effect" and to ensure the bound of the relative error for the numerical solution of the problem (4)–(6), it is necessary to impose the condition

$$k^{\beta}(kh)^{\gamma} = constant. \tag{7}$$

Here, h denotes the step size, and $\beta > 0$, $\gamma > 0$ are real numbers. For example, β=2, and γ=2 and 4, when the solution is computed by the standard central finite difference scheme and the compact fourth order difference scheme, respectively. The "pollution effect" cannot be eliminated by using the finite element methods in higher dimension. Considering that the Helmholtz problem is solved with fixed values of k and h, and due to the relation given in (7), the numerical error will be increasing even when the step size is reduced. This adverse behavior is the direct consequence of the "pollution effect", and more detailed discussion is reported in [31].

Babuška and Sauter [3] reported that the "pollution effect" cannot be avoided on a general bounded domain in 2D or 3D. It should be noted that the standard finite difference and the high order compact methods are constructed based on a truncated Taylor series expansion, and the truncation errors depend on the wave numbers and thus cause unavoidable "pollution effect". To remove the pollution, some pollution-free difference schemes for the one-dimensional Helmholtz equation are proposed in [16,22,32,36], and the derivation takes account of all terms in the Taylor series expansion and other techniques. An important feature of a pollution-free difference scheme is that unlike (7), the resulting numerical error depends only on the mesh size h and independent of the wave number k. Therefore, the numerical error will be decreasing as the step size is reduced, and the details are reported in [32,36]. Consequently, we only require to satisfy the "rule of thumb" to capture the wave profile of the solution, which suggests that the condition $kh = constant$ is needed compared to (7). The common "rule of thumb" refers to taking $kh \approx 0.6$ to ensure the wave profile is recovered by at least 10 discrete points in each wave length. For highly oscillatory wave solutions, a larger value of kh could be employed. The computational simulations confirm that the stability and accuracy for the pollution-free difference schemes are not affected even when $kh > 1$. In contrast, $kh \ll 1$ is usually required when the standard finite difference or compact difference schemes are employed for problems at high wave numbers. However, when extending directly this idea to the problems on a general domain in 2D and 3D, the term u_{xxyy} will appear in the Taylor series which cannot be eliminated exactly. The classical numerical differentiation approximation is unavoidable when constructing a numerical scheme, thus the "pollution effect" will be generated in the scheme. Initial results have been obtained on certain special computational domains. By a separation of variables technique, the new approach has been extended to problems on annulus and hollow sphere domains [34], on circular cylindrical domains [15] and on rectangle domains [33].

In this paper, we continue our work on developing accurate and efficient difference schemes for solving the Helmholtz equation (4)–(6) on the disk and sphere domains. Different from the problems on the domains considered in the previous study reported in [32], the singularity exists in the reduced one-dimensional problem in the neighborhood of the origin. We will confirm that the "pollution effect" cannot be avoided only on a very small neighborhood of 0, but it is possible to employ a pollution-free difference scheme on the rest of the domain away from 0. Based on the present study, a local mesh refinement difference scheme is proposed in Section 2, and the numerical solution is computed using two different grids. It is demonstrated that compared to using a uniformly fine grid for the entire computational domain, the proposed scheme is capable of keeping the same numerical accuracy but with remarkable saving in the computational storage. Section 3 presents a two-level local refinement so that the overall accuracy can be further improved. The superior performance of the proposed difference schemes is verified by numerical simulations. Finally, conclusions will be given in Section 4.

2. Difference schemes

We now consider the Helmholtz equation in the polar and spherical coordinates. For more details on this topic, the reader is referred to [7,19,24,26,27,34,35]. This section consists of several parts. First, we introduce the model in 2D and 3D