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# Numerical solution of nonlinear weakly singular partial integro-differential equation via operational matrices

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ABSTRACT

In this paper, we propose and analyze an efficient matrix method based on shifted Legendre polynomials for the solution of non-linear volterra singular partial integro-differential equations(PIDEs). The operational matrices of integration, differentiation and product are used to reduce the solution of volterra singular PIDEs to the system of non-linear algebraic equations. Some useful results concerning the convergence and error estimates associated to the suggested scheme are presented. illustrative examples are provided to show the effectiveness and accuracy of proposed numerical method.



## 1. Introduction

Volterra type integral equations with weakly singular kernel appears in many problems of mathematical physics and chemical reaction, such as theory of elasticity, hydrodynamics, heat conduction, stereology [1], the radiation of heat from a semi infinite solids [2] and many other applications. Equations of this type have been studied by several authors [3–11].

In this paper we study second kind Volterra singular PIDE of the form

$$\phi_t = \phi(x,t) + g(x,t) + \int_0^x \int_0^t \frac{G\phi(\zeta,\eta)}{(x-\zeta)^\alpha} d\eta d\zeta \qquad \begin{aligned} &0 \le \zeta \le x,\ 0 \le \eta \le t \\ &(x,y) \in [0,1]\times[0,1] \\ &0 < \alpha < 1 \end{aligned} \tag{1}$$

with initial condition $\phi(x,0) = \phi_0(x)$.

Where, $\phi$ is unknown function in $\Lambda(=[0,1]\times[0,1])$ which should be determined and the functions $g$, $\phi_0(x)$ are known. $G$ is a non-linear operator. The functions $g(x, t)$ and $\phi(x, t)$ are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $\phi \in C(\Lambda)$. We assume that the non-linear term $G\phi$ satisfies the Lipschitz condition in $L^2(\Lambda)$. It is clear that in the above equation the kernel function has a weak singularity at the origin.

Problems involving PIDEs arise in fluid dynamics, engineering, mathematical biology and other areas. In general, PIDEs are difficult to solve analytically. Main challenges in solving PIDEs analytically are due to many factors, such as non-linearity, non-local phenomena, multi-dimensionality. Various numerical techniques have been developed for the solution of PIDEs

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[12–19]. The solution of the above problem in the case of linear operator has already been discussed by Singh et al. [19]. In the conclusion of the paper [19], author had mentioned the future scope of the non-linearity case.

In this paper, we have used the 2*D* shifted orthogonal Legendre polynomial approximation to find the numerical solution of Eq. (1). The advantage of orthogonal system, proposed in this paper is that the Legendre bivariate polynomial provide an accurate approximation of the problem solution with reduced number of basis functions.

The remainder of the paper is organized as follows; we give a brief review of Legendre polynomial, their properties and function approximation in Section 2. Section 3 is devoted to the operational matrices. In this section we have derived the Legendre operational matrix of integration, derivative matrix and the product operational matrix. In Section 4 we discussed two types of problems for the cases $G\phi(x,t)=[\phi(x,t)]^p$ and $G\phi=[\gamma_{i,j}\phi(x,t)]^p$, where, $\gamma_{i,j}$ is a differential operator, $p$ is a positive integer, and how the operational matrices can be used to reduce the problem (1) into a set of non-linear algebraic equations for both the cases is also explained. Convergence analysis and error estimates for our proposed method are derived in Sections 5 and 6. In Section 7 we demonstrate the accuracy of the proposed method by considering several test functions. Finally, some concluding remarks are given in Section 8.

## 2. Properties of 2*D* shifted Legendre polynomials

### 2.1. Definition of 2D shifted Legendre polynomials

2*D* shifted Legendre polynomials are defined on $\Lambda$ as follows:

$$\psi_{rs}(x,t)=P_r(2x-1)P_s(2t-1), \qquad r,s=0,1,2,\dots \tag{2}$$

where, $P_r(x)$ and $P_s(x)$ are the Legendre polynomials, of order $r$ and $s$, respectively defined on the interval $[-1,1]$ and satisfy the following recursive formula

$$P_0(x)=1, P_1(x)=x \quad \text{and} \quad P_{s+1}(x)=\frac{2s+1}{s+1}xP_s(x)-\frac{s}{s+1}P_{s-1}(x),\ \ s=1,2,3,\dots$$

They form a complete basis over the interval [−1, 1]. 2*D* shifted Legendre polynomials are orthogonal with respect to weight function $\omega$(x, t) such that

$$\int_0^1\int_0^1 \omega(x,t)\psi_{rs}(x,t)\psi_{ij}(x,t)dxdt=\begin{cases}\dfrac{1}{(2r+1)(2s+1)}, & \text{for } i=r, j=s,\\ 0, \ \text{otherwise}\end{cases} \tag{3}$$

Let $X=L^2(\Lambda)$ be the inner product space. Then the inner product in this space is defined by

$$\langle u(x,t),v(x,t)\rangle=\int_0^1\int_0^1 u(x,t)v(x,t)dxdt$$

and the norm is defined as follows

$$\| u(x,t)\|_2=\langle u(x,t),u(x,t)\rangle^{1/2}=\left(\int_0^x\int_0^t |u(x,t)|^2\,dxdt\right)^{1/2}$$

### 2.2. Function approximation

Let g(*x*, *t*) be an arbitrary function in $L^2(\Lambda)$, then it can be approximated as

$$g(x,t)\simeq\sum_{r=0}^{N}\sum_{s=0}^{N}F_{rs}\psi_{rs}(x,t)=F^T\Psi(x,t), \tag{4}$$

where, $F$ and $\Psi$ are $(N+1)^2\times 1$ vector given by

$$F=[F_{00},F_{01},\dots,F_{0N},F_{10},F_{11},\dots,F_{1N},\dots,F_{N0},\dots,F_{NN}]^T, \tag{5}$$

$$\Psi(x,t)=[\psi_{00}(x,t),\psi_{01}(x,t),\dots,\psi_{0N}(x,t),\psi_{10}(x,t),\psi_{11}(x,t),\dots,\psi_{1N}(x,t),\dots,\psi_{N0}(x,t),\dots,\psi_{NN}(x,t)]^T \tag{6}$$

and, $\psi_{rs}(x,t)=\psi_r(x)\psi_s(t)$.