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Fractional differential equations with a constant delay: Stability and asymptotics of solutions

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ABSTRACT

The paper discusses stability and asymptotic properties of a fractional-order differential equation involving both delayed as well as non-delayed terms. As the main results, explicit necessary and sufficient conditions guaranteeing asymptotic stability of the zero solution are presented, including asymptotic formulae for all solutions. The studied equation represents a basic test equation for numerical analysis of delay differential equations of fractional type. Therefore, the knowledge of optimal stability conditions is crucial, among others, for numerical stability investigations of such equations. Theoretical conclusions are supported by comments and comparisons distinguishing behaviour of a fractional-order delay equation from its integer-order pattern.

1. Introduction

We investigate stability and asymptotic properties of the fractional delay differential equation

$$\mathrm{D}^{\alpha} y(t) = a y(t) + b y(t-\tau), \qquad t > 0 \tag{1}$$

with real coefficients a, b, a positive real lag τ and the fractional Caputo derivative operator D^{α} ($0 < \alpha < 1$ is assumed to be a real number).

Letting $\alpha \to 1$ from the left, $\mathrm{D}^{\alpha} y(t)$ becomes $y'(t)$ and (1) is reduced to the classical delay differential equation

$$y'(t) = a y(t) + b y(t-\tau), \qquad t > 0, \tag{2}$$

studied frequently due to its theoretical as well as practical importance (see, e.g. [12]). This equation serves, among others, as the basic test equation for stability analysis of various numerical discretizations of delay differential equations (see, e.g. [1,9]). In this connection, stability conditions for (2) are traditionally required in the optimal form, i.e. as the necessary and sufficient ones. There are known two types of such conditions for asymptotic stability of the zero solution of (2) that we recall in the following two assertions (see [12]). As it is customary, by asymptotic stability of the zero solution of (2) we understand the property that any solution y of (2) is eventually tending to the zero solution.

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Theorem 1. *Let a, b and $\tau > 0$ be real numbers. The zero solution of (2) is asymptotically stable if and only if the couple (a, b) is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve*

$$a = \varphi \cot(\tau\varphi), \quad b = -\frac{\varphi}{\sin(\tau\varphi)}, \qquad \varphi \in \Big(0, \frac{\pi}{\tau}\Big)$$

from below.

Theorem 2. *Let a, b and $\tau > 0$ be real numbers. The zero solution of (2) is asymptotically stable if and only if either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary}, \tag{3}$$

or

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}. \tag{4}$$

While Theorem 1 describes the stability boundary for (2) in the (a, b)-plane, the conditions of Theorem 2 seem to be more explicit. In particular, $(4)_2$ presents the value of the stability switch when (2) loses its stability property as the delay τ is monotonically increasing.

It is also well-known that asymptotic stability of the zero solution of (2), described via the conditions of Theorems 1 and 2, is of exponential type, i.e. there exists $\delta > 0$ such that $y(t) = \mathcal{O}(\exp[-\delta t])$ as $t \to \infty$ for any solution y of (2). The value δ depends on location of roots of the characteristic equation

$$s - a - b\exp[-s\tau] = 0 \tag{5}$$

with respect to the imaginary axis (its estimates are discussed, e.g. in [8]).

The involvement of fractional-order derivatives into delay differential equations represents a new type combining advantages of both delayed and non-integer derivative terms, especially hereditary properties, more degrees of freedom and other advantages of fractional modelling. Since application areas of fractional delay differential equations are especially control theory and robotics, the question of their stability (and asymptotics) is again of main interest. In general, stability and asymptotic analysis of fractional delay differential equations is just at the beginning. As it is evident from the literature (see, e.g. [6,16,17,21]), almost all the existing stability results on autonomous equations of this type are based on the root locus of appropriate characteristic equations, and they do not provide universally acceptable effective criteria for testing stability of a given fractional delay equation.

Therefore, the main goal of this paper is to extend the above stated properties of (2) to (1). Since (1) may serve as a basic prototype of fractional delay differential equations, formulation of non-improvable stability conditions and related asymptotic formulae is of a great importance in qualitative as well as numerical analysis of fractional delay differential equations.

Following the classical case, we say that the zero solution of (1) is asymptotically stable if any solution y of (1) is eventually tending to the zero solution. Similarly, the zero solution of (1) is called stable if any its solution is eventually bounded. To describe asymptotics of solutions of (1), we shall use the following asymptotic symbols:

$$f(t) \sim g(t) \qquad \text{if} \qquad \lim_{t\to\infty} \frac{|f(t)|}{|g(t)|} = K > 0,$$

$$f(t) \sim_{sup} g(t) \qquad \text{if} \qquad \limsup_{t\to\infty} \frac{|f(t)|}{|g(t)|} = K > 0.$$

We note that (1) has been studied in the purely delayed case (when $a = 0$) in [5,15]. While the first paper presents stability criterion based on a transcendent inequality involving the fractional Lambert function, the latter paper already contains explicit conditions (which are, as we have already mentioned, very rare for fractional delay differential equations). We recall here the explicit relevant results (reformulated to our notation) characterizing stability and asymptotics of (1) with $a = 0$ (see [15, Theorem 5.1]).

Theorem 3. *Let $0 < \alpha < 1$, $b \neq 0$ and $\tau > 0$ be real numbers.*

(i) *The zero solution of*

$$\mathrm{D}^\alpha y(t) = b y(t - \tau), \qquad t > 0 \tag{6}$$

is asymptotically stable if and only if

$$-\Big(\frac{\pi - \alpha\pi/2}{\tau}\Big)^\alpha < b < 0.$$

In this case, $y(t) \sim t^{-\alpha}$ as $t \to \infty$ for any solution y of (6).

(ii) *The zero solution of (6) is stable, but not asymptotically stable, if and only if*

$$b = -\Big(\frac{\pi - \alpha\pi/2}{\tau}\Big)^\alpha.$$