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A novel unified approach to invariance conditions for a linear dynamical system

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ABSTRACT

In this paper, we propose a novel, simple, and unified approach to explore sufficient and necessary conditions, i.e., invariance conditions, under which four classic families of convex sets, namely, polyhedra, polyhedral cones, ellipsoids, and Lorenz cones, are invariant sets for a linear discrete or continuous dynamical system.

For discrete dynamical systems, we use the Theorems of Alternatives, i.e., Farkas lemma and *S*-lemma, to obtain simple and general proofs to derive invariance conditions. This novel method establishes a solid connection between optimization theory and dynamical system. Also, using the *S*-lemma allows us to extend invariance conditions to any set represented by a quadratic inequality. Such sets include nonconvex and unbounded sets.

For continuous dynamical systems, we use the forward or backward Euler method to obtain the corresponding discrete dynamical systems while preserves invariance. This enables us to develop a novel and elementary method to derive invariance conditions for continuous dynamical systems by using the ones for the corresponding discrete systems.

Finally, some numerical examples are presented to illustrate these invariance conditions.

1. Introduction

Positively invariant sets play a key role in the theory and applications of dynamical system s. Stability, control and preservation of constraints of dynamical system s can be formulated, somehow in a geometrical way, with the help of positively invariant sets. For a given dynamical system, both of continuous or discrete time, a subset of the state space is called positively invariant set for the dynamical system if containing the system state at a certain time then forward in time all the states remain within the positively invariant set. Geometrically, the trajectories cannot escape from a positively invariant set if the initial state belongs to the set. The dynamical system is often a controlled system of which the maximal (or minimal) positively invariant set is to be constructed.

It is well known, see e.g., Blanchini [9], Blanchini and Miani [12], and Polanski [42], that the Lyapunov stability theory is used as a powerful tool in obtaining many important results in control theory. The basic framework of the Lyapunov stability theory synthesizes the identification and computation of a Lyapunov function of a dynamical system. Usually positive definite quadratic functions serve as candidate Lyapunov functions. Sufficient and necessary conditions for positive invariance of a polyhedral set with respect to discrete dynamical systems were first proposed by Bitsoris [6,7]. A novel positively

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invariant polyhedral cone was constructed by Horváth [32]. The Riccati equation was proved to be connected with ellipsoidal sets as invariant sets of linear dynamical systems, see e.g., Lin et al. [37] and Zhou et al. [61]. Birkhoff [5] proposed a necessary condition for positive invariance on a convex cone for linear discrete system. A sufficient and necessary condition for positive invariance on a nontrivial convex set for linear discrete systems was derived by Elsner [18]. Stern [49] studied the properties of positive invariance on a proper cone for linear continuous systems. For a more general case, the mapping from a polyhedral cone to another polyhedral cone was studied by Haynsworth et al. [27], and the mapping from a convex cone to another convex cone in finite-dimensional spaces was studied by Tam [52,53]. Here we note that when the two cones are the same, then this is equivalent to positive invariance for discrete system. The concept of cross positive matrices, which was introduced by Schneider and Vidyasagar [46], are used as tools to prove positive invariance of a Lorenz cone by Loewy and Schneider [38]. According to Nagumo's theorem [41] and the theory of cross positive matrices, Stern and Wolkowicz [50] presented sufficient and necessary conditions for a Lorenz cone to be positively invariant with respect to a linear continuous system. A novel proof of the spectral characterization of real matrices that leave a polyhedral cone invariant was proposed by Valcher and Farina [56]. The spectral properties of the matrices, e.g., theorems of Perron–Frobenius type, were connected to set positive invariance by Vandergraft [46]. Recently, the discrete system has been extended to the case when the state variable belongs to the tangent bundle of a Riemannian manifold or a Lie algebra by Fiori, see, [20,21]. The problem of the unconditional invariance is posed for the first time in the history of control theory by Shipanov [47]. Gusev and Likhtarnikov [26] present a survey of the history of two fundamental results of the mathematical system theory – the Kalman–Popov–Yakubovich lemma and the theorem of losslessness of the *S*-procedure. For an excellent book about the *S*-procedure the reader is referred to [1] by Aizerman and Gantmacher. An extension of invariance conditions to nonlinear dynamical system can be found in [36].

Mathematical modeling of many problems from the real world often leads to differential equations in continuous form. When we solve these differential equations numerically, we not only need to obtain a good approximation of the differential equations, but also hope to preserve the basic characteristics of these mathematical variables and models. Invariance preserving is one of the latter type requirements. In fact, there are various characteristics preserving topics, e.g., positivity preserving, strong stability preserving, area preserving, etc, which are extensively studied in recent decades. *(1). Positivity Preserving:* Positivity preserving is an important topic in the numerical analysis community, see, e.g., [32,33,58–60]. Positivity preserving is equivalent to invariance preserving in the positive orthant, i.e., consider the positive orthant, which is a polyhedral cone. Let us assume that the positive orthant is an invariant set for a continuous system, and assume that it is also an invariant set for the discrete system which is obtained by using a discretization method with a certain steplength. In practice, many variables, e.g., energy, density, mass, etc, are nonnegative. When these variables are used in some mathematical models in a continuous form, e.g., in the heat equation, one should choose appropriate discretization method with appropriate steplength such that solution of the discretized systems are also nonnegative. *(2). Strong Stability Preserving (SSP):* Strong stability preserving (SSP) numerical methods are developed to solve ordinary differential equations, see, e.g., [23,24], etc. Particularly, SSP numerical method are used for the time integration of semi-discretizations of hyperbolic conservation laws. It is well known that the exact solutions of scalar conservation laws holds the property that total variation does not increase in time, see, e.g., [24]. SSP methods are also referred to as total variation diminishing methods. These are higher order numerical methods that also preserve this property. *(3). Area Preserving-Symplectic Methods:* Intuitively, a map from the phase-plane to itself is said to be symplectic if it preserves areas. In mathematics, a matrix $M \in \mathbb{R}^{2n\times 2n}$ is called symplectic if it satisfies the condition $M^T\Omega M = \Omega$, where $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A symplectic map is a real-linear map T that preserves a symplectic form f, i.e., $f(Tx, Ty) = f(x, y)$ for all x, y, see, e.g., [40]. A numerical one-step method $x_{n+1} = D_{\Delta t}(x_n)$ is called symplectic if, when applied to a Hamiltonian system, the discrete flow $x \to D_{\Delta t}(x)$ is a symplectic map for all sufficiently small step sizes, see, e.g., [19,39], etc. There is one compelling example that shows symplectic methods are the right way to solve planetary trajectories. If we solve the trajectory of the earth using forward Euler method, then the discrete trajectory will spiral away from the sun. If we use backward Euler method, then the discrete trajectory will sink into the sun. If we use symplectic methods, then the discrete trajectory will stay on the original continuous trajectory.

In many applications, the models are represented as a partial differential equation (PDE), e.g., heat equation, then certain numerical methods, e.g., finite difference methods, finite element methods, etc., may be first applied to the spatial variable to obtain a ODE (dynamical system). The numerical methods for ODE are then used to obtain the discrete form of the model. Therefore, invariance condition for a ODE (dynamical system) is crucial for models even within a PDE form. We point out that the invariance condition for the numerical methods for the spatial variable of the PDE is an important research topic but out of the scope of this paper.

In this paper we deal with dynamical systems in finite dimensional spaces and introduce a novel and unified method for the determination of whether a set is a positively invariant set for a linear dynamical system. Here the sets are ellipsoids, polyhedral sets or - not necessarily convex - second order sets including Lorenz cones. In addition, we formulate optimization methods to check the resulting equivalent conditions.

The main tool in the continuous time case consists of the explicit computation of the tangent cones of the positively invariant sets and their application along the lines of the Nagumo theorem [41]. This theorem says that a set is positively invariant, under some conditions on solvability of the underlying differential equation, if and only if at each point of the set, the vector field of the differential equation points toward the tangent cone at that point. The resulting conditions are constructive in the sense that they can be checked by well established optimization methods. Our unified approach is based