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# Optimal consumption and portfolio selection problems under loss aversion with downside consumption constraints

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ABSTRACT

This paper investigates continuous-time optimal portfolio and consumption problems under loss aversion in an infinite horizon. The investor's goal is to choose optimal portfolio and consumption policies to maximize total discounted S-shaped utility from consumption. The consumption rate process is subject to a downside constraint. The optimal consumption and portfolio policies are obtained through the martingale method and replication technique. Numerical results indicate the differences between the loss averse investor and the constant relative risk averse (CRRA) investor on the optimal consumption and portfolio policies: the loss averse investor likes consuming more money but exposing less to risk than that of the CRRA investor, and the optimal wealth, as a function of state price density, drops faster for the CRRA investor than that for the loss averse investor.



## 1. Introduction

There have been many research works on financial portfolio selection models since Merton [10] started this research in 1969. The traditional financial portfolio selection models are based on the assumption that investors are rational and risk averse. However, in the real world, substantial experimental evidences are inconsistent with this rationality hypothesis. Kahneman and Tversky [8,17] found that investors treat gains and losses differently, they are risk-averse on gains and risk-seeking on losses, and more sensitive to losses than to gains, and what is more, the investors overweigh small probabilities and underweigh large probabilities. Because of these things, Kahneman and Tversky [8] for the first time proposed the concept of loss aversion in the framework of prospect theory, and later Tversky and Kahneman [16] defined it for choice under uncertainty . Loss aversion is an important psychological concept. In mathematics, it can be represented by an utility function which is concave for gains and convex for losses, and steeper for losses than for gains. Loss aversion can explain many phenomena such as the endowment effect (Thaler [15]), the status quo bias (Samuelson and Zeckhauser [12]), and the equity premium puzzle (Benartzi and Thaler [2]), which remain paradoxes in traditional selection theory. Therefore, it has frequently been applied in behavioral finance and received more and more attention. Nevertheless, as far as we know, there are very few literatures available that investigated continuous-time portfolio optimization problems under loss aversion although it has been proposed for ages.

Berkelara, et al. [3] derived closed-form solutions for the optimal portfolio choice under loss aversion by extending the martingale methodology to allow for pseudoconcave utility functions. Jin and Zhou [6] established a general continuous-time

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portfolio selection model under the cumulated prospect theory (CPT) for the first time, and they developed a systematic approach to overcome the immense difficulties which arise from the analytically ill-behaved utility functions and probability distortions. Zhang, et al. [19] formulated a continuous-time behavioral portfolio selection model where the losses were constrained by a prespecified upper bound and derived the optimal solution explicitly by solving a concave Choquet minimization problem with an additional upper bound. Mi, et al. [11] investigated a dynamic asset allocation model for loss-averse investors in a jump-diffusion financial market, where the optimal wealth process and investment strategies were obtained using martingale approach. Guo [5] studied an optimal portfolio selection problem for the insurer with loss aversion. Fulga [4] presented an approach which incorporates loss aversion preferences in the Mean-Risk framework. However, all of them supposed that the term of investment was finite and the investor was only concerned with the terminal wealth and did not care about the consumption. We investigate the optimal portfolio and consumption problem under loss aversion in an infinite horizon. We need not consider the terminal wealth downside constraint since we only consider an infinite horizon case. However, compared with a finite horizon, optimal consumption and portfolio selection problem in an infinite horizon requires some additional technicalities because the convergence of infinite integral has to be discussed.

In standard consumption and investment models where investors are assumed to behave rationally and maximize expected utility, Zariphopoulou [18] examined a general investment and consumption problem for a single agent who consumes and invests in a riskless asset and a risky one, and with binding trading constraints, limited borrowing, and no bankruptcy, she proved that the value function was the unique smooth solution to the associated Hamilton–Jacobi–Bellman equation and provided the optimal consumption and portfolio policies in feedback form. Lakner and Nygren [9] solved the portfolio optimization problem with both consumption and terminal wealth downside constraints using the gradient operator and the Clark–Ocone formula in Malliavin calculus on a finite horizon. Shin, et al. [13] studied a general optimal consumption and portfolio selection problem of an infinitely-lived investor whose consumption rate process is subjected to downside constraint. Supposing that consumption can never fall below a fixed proportion of the running maximum of past consumption, Arun [1] solved the Merton problem. Therefore, based on these studies, we assume the consumption rate process of a loss averse investor is subjected to a downside constraint. We derive the optimal portfolio and consumption policies in explicit forms. Moreover, we compare the optimal consumption and portfolio policies for loss averse investor with the case of CRRA investor by some numerical results to demonstrate the influence of loss aversion.

The remainder of this paper is organized as follows. In Section 2, we formulate the optimal portfolio and consumption model under loss aversion. Section 3 solves the optimal portfolio and consumption problem with the downside consumption constraints, and gives the closed-form solutions for the portfolio and consumption policies. Section 4 presents some numerical results to compare the optimal consumption and portfolio policies for loss averse investor with the case for CRRA investor. Finally, Section 5 concludes this work.

## 2. Problem formulation

We consider an infinite horizon $t \in (0, \infty)$. Let $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space. On this space we define an n-dimensional standard $\mathcal{F}_t$-adapted Brownian motion $W_t = (W_t^1, \ldots, W_t^n)^T$ with $W_0 = 0$, where the filtration $\mathcal{F}_t = \sigma\{W_\tau : 0 \leq \tau \leq t\}$. Here and throughout this paper $A^T$ denotes the transpose of matrix $A$.

Suppose that there is a frictionless market in which $n+1$ assets are traded continuously. One of the assets $S_0(t)$ is riskless (for example: bank account). Here we suppose its dynamics is governed by:

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = s_0, \tag{1}$$

The remaining assets are risky (for example: stocks), their prices $S_i(t),\ i = 1, \ldots, n,$ are modeled by Ito processes:

$$dS_i(t) = S_i(t)\left[\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_t^i\right],\ S_i(0) = s_i,\ i = 1, \ldots, n, \tag{2}$$

where the risk-free rate process $r$, $n$-dimensional mean rate of return process $\mu(\cdot) = (\mu_1, \ldots, \mu_n)^T$ and $n \times n$-matrix-valued volatility process $\sigma = (\sigma_{ij})_{n\times n}$ are constant.

It is well known that a complete market implies the existence and uniqueness of a state price density $\xi_t$, given by

$$\xi_t = \exp\big(-(r+\gamma)t - \kappa^T W_t\big),$$

where $\kappa = \sigma^{-1}(\mu - rI)$ denotes the market prices of the risky assets, and $\gamma = \frac{1}{2}\|\kappa\|^2$. We can rewrite the state price density process as

$$d\xi_t = -\xi_t(rdt + \kappa^T dW_t), \quad \xi_0 = 1.$$

Define $\widetilde{\xi}_t^y = ye^{\rho t}\xi_t$ for $y > 0$, then

$$d\widetilde{\xi}_t^y = \widetilde{\xi}_t^y[(\rho - r)dt + \kappa^T dW_t], \quad \widetilde{\xi}_0 = y.$$

Suppose that $X_t$ is the total wealth of the agent at time $t$. We require $X_t \geq 0$ for $t \in (0, +\infty)$. Let $\pi_t = (\pi_t^1, \ldots, \pi_t^n)^T$ be the fraction invested on the risky assets and $C_t$ be the consumption rate at time $t$. Thus, the fraction of $1 - \sum_{i=1}^n \pi_t^i$ will be invested on the riskless asset. Then, given initial wealth $x_0 > 0$, the wealth process can be written as:

$$dX_t = \Big(X_t\big[r + (\mu - rI)^T \pi_t\big] - C_t\Big)dt + X_t \pi_t^T \sigma dW_t. \tag{3}$$